Math Camp 2025 – Problem Set 9

Read the following problems carefully and justify all your work. Avoid using calculators or computers. (This was adapted almost verbatim from a problem set by Christopher Lucas.)

1. There are 10 first-year students in the Department of Political Science at Bear University. The Director of Graduate Studies (DGS) asks each student to select one from four different lunch seminars to attend. The DGS records how many students choose each lunch seminar. She obtains four numbers x_1, x_2, x_3, x_4, e.g., she could obtain $1, 2, 0, 7$. How many different possibilities exist? Explain your answer.

Comment. The trick is to view the possibilities as distinct permutations of the word `X|XX||XXXXXXX`. Convince yourself that this is the case and count them.

Answer. We look for the number of lists of four nonnegative integers (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 10$. The trick is to notice that we can see this as distinct permutations of the word `X|XX||XXXXXXX` which has ten Xs and three bars. The Xs represent students, and the bars divide them between the different seminars. For example, `X|XX||XXXXXXX` corresponds to $(1, 2, 0, 7)$, and `XXX|XXX|XXXX|` corresponds to $(3, 3, 4, 0)$. Choosing a permutation of `X|XX||XXXXXXX` is the same as choosing where to put the three bars; the Xs then go in the rest of the spaces. There are $10 + 3 = 13$ places to put the bars, and three bars, so the number is $\binom{13}{3} = 286$.

2. A box contains three coins. One coin has heads on both sides, one has tails on both sides, and one has one side with head and the other side with tail. One coin is randomly selected from the box, and you observe that one random side of the selected coin is head. What is the probability that the other side of the selected coin is also head? Explain your answer.

Answer. We want to calculate Pr(Coin is HH | A Side is H). We have

$$\Pr(\text{Coin is HH} \mid \text{A Side is H}) = \frac{\Pr(\text{Coin is HH and A Side is H})}{\Pr(\text{A Side is H})} = \frac{\Pr(\text{Coin is HH})}{\Pr(\text{A Side is H})}.$$

Now, $\Pr(\text{Coin is HH}) = \frac{1}{3}$, and Pr(A Side is H) can be calculated by the law of total probability:

$$\begin{aligned}
\Pr(\text{A Side is H}) &= \Pr(\text{A Side is H and Coin is HH}) + \Pr(\text{A Side is H and Coin is HT}) \\
&= \Pr(\text{A Side is H} \mid \text{Coin is HH})\Pr(\text{Coin is HH}) \\
&\quad + \Pr(\text{A Side is H} \mid \text{Coin is HT})\Pr(\text{Coin is HT}) \\
&= 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{2}.
\end{aligned}$$

Therefore,

$$\Pr(\text{Coin is HH} \mid \text{A Side is H}) = \frac{\Pr(\text{Coin is HH})}{\Pr(\text{A Side is H})} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

3. Suppose that a district contains 100 likely voters, of which 60% support candidate 1. You randomly sample a subset of k likely voters without replacement (this means taking a random subset of k voters), and ask about their preferences.

(a) If $k = 20$, what is the probability that exactly 60% of your sample are supporters of candidate 1?

(b) If $k = 20$, what is the probability that you correctly identify the leading candidate (i.e., that at least 50% of your sample are supporters of candidate 1)? Show how this probability can be calculated.

Answer.

(a) We want the probability that $60\% \times 20 = 12$ voters in the sample support candidate 1. There are $60\% \times 100 = 60$ candidate-1 supporters in the population. How many subsets of 20 people contain 12 candidate-1 supporters? To form that subset we need to choose 12 candidate-1 supporters among the 60 in the population, and then choose $20 - 12 = 8$ non-supporters among the 40 in the population. So, there are $\binom{60}{12}\binom{40}{8}$ possibilities. Each one is equally likely, and in total there are $\binom{100}{20}$ samples. Therefore the probability is

$$\frac{\binom{60}{12}\binom{40}{8}}{\binom{100}{20}} \approx 20\%.$$

(b) We want the probability that at least $50\% \times 20 = 10$ voters in the sample support candidate 1. This is

$$\sum_{s=10}^{20} \frac{\binom{60}{s}\binom{40}{20-s}}{\binom{100}{20}} \approx 90\%.$$

4. The United States Senate contains two senators from each of the 50 states.

(a) If a committee of 50 senators are selected at random, what is the probability that the group will contain one senator from each state?

(b) If a committee of k senators is selected at random, what is the probability that it will contain at least one of the two senators from New Jersey?

(c) What are the possible sizes (number of members) of the committee such that there is at least 50% probability that the committee will contain at least one of two senators from New Jersey?

Answer.

(a) There are $\binom{100}{50}$ committees. We want to find the number of committees that have exactly one senator from each state. We are choosing one senator from each state, and there are 2 for each state, so the number of choices is 2^{50}. The answer is then

$$\frac{2^{50}}{\binom{100}{50}},$$

which is a tiny number.

(b) There are $\binom{100}{k}$ committees. If the committee contains one senator from New Jersey, there are 2 choices for the senator, and then $\binom{98}{k-1}$ for the rest. If the committee contains both senators from New Jersey, there are $\binom{98}{k-2}$ choices for the rest. In sum, the probability is

$$\frac{2\binom{98}{k-1}+\binom{98}{k-2}}{\binom{100}{k}}.$$

(c) We have

$$\begin{aligned}\frac{2\binom{98}{k-1}+\binom{98}{k-2}}{\binom{100}{k}} &= \left[2\frac{98!}{(k-1)!(98-k+1)!}+\frac{98!}{(k-2)!(98-k+2)!}\right]\frac{k!(100-k)!}{100!}\\ &= \left[2\frac{k!(100-k)!}{(k-1)!(99-k)!}+\frac{k!(100-k)!}{(k-2)!(100-k)!}\right]\frac{1}{100\cdot 99}\\ &= [2k(100-k)+k(k-1)]\frac{1}{100\cdot 99}\\ &= \frac{k(199-k)}{100\cdot 99}.\end{aligned}$$

We want this to be at least 50%, i.e., $\frac{k(199-k)}{100\cdot 99} \geqslant \frac{1}{2}$, or $k(199-k) \geqslant 50\cdot 99$, i.e., $k^2-199k+4950 \leqslant 0$. We can complete squares: $k^2-199k+4950 = (k-\frac{199}{2})^2-\frac{199^2}{4}+4950$, so $k^2-199k+4950 \leqslant 0$ iff $|k-\frac{199}{2}| \leqslant \frac{1}{2}\sqrt{19801}$, i.e., k is between $\frac{199}{2}-\frac{1}{2}\sqrt{19801} \approx 29.14$ and $\frac{199}{2}+\frac{1}{2}\sqrt{19801} \approx 169.86$. The possible k are the integers $30, 31, \ldots, 169$.

5. The *odds* of an event with probability p are defined to be $\frac{p}{1-p}$, e.g., an event with probability $3/4$ is said to have odds of 3 to 1 in favor (or 1 to 3 against). We are interested in a hypothesis H (which we think of as an event), and we gather new data as evidence (expressed as an event D) to study the hypothesis. The *prior* probability of H is our belief of the probability for H being true before we gather the new data; the *posterior* probability of H is our belief of the probability for H being true after we gather the new data. The *likelihood ratio* is defined as $\dfrac{\Pr(D|H)}{\Pr(D|H^c)}$.

Show that Bayes' rule can be expressed in terms of odds as follows: *the posterior odds of a hypothesis H are the prior odds of H times the likelihood ratio*.

Answer. The posterior odds of a hypothesis H is

$$\frac{\Pr(H|D)}{1-\Pr(H|D)} = \frac{\Pr(H|D)}{\Pr(H^c|D)} = \frac{\dfrac{\Pr(D|H)\Pr(H)}{\Pr(D)}}{\dfrac{\Pr(D|H^c)\Pr(H^c)}{\Pr(D)}} = \frac{\Pr(D|H)\Pr(H)}{\Pr(D|H^c)\Pr(H^c)}$$
$$= \frac{\Pr(H)}{1-\Pr(H)}\frac{\Pr(D|H)}{\Pr(D|H^c)},$$

which is the prior odds of H times the likelihood ratio.

6. Prove the following statements. For each statement, explain what it means in practice with your own example.

(a) $\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B)$.

(b) If $\Pr(A) > 0, \Pr(B) > 0$, and $\Pr(A) < \Pr(A|B)$, then $\Pr(B) < \Pr(B|A)$.

Answer.

(a) We have $A \cup B = A \cup (B \setminus A)$, and $A \cap (B \setminus A) = \varnothing$, so $\Pr(A \cup B) = \Pr(A \cup (B \setminus A)) = \Pr(A) + \Pr(B \setminus A)$. Now, $B \setminus A \subset B$, so $\Pr(B \setminus A) \leqslant \Pr(B)$, and $\Pr(A \cup B) \leqslant \Pr(A) + \Pr(B)$, as desired.

(b) We have $\Pr(A) < \Pr(A \cap B)/\Pr(B)$, so $\Pr(B) < \Pr(A \cap B)/\Pr(A) = \Pr(B|A)$, as desired.

7. Suppose that the PDF of a random variable X is as follows:

$$f(x) = \begin{cases} \frac{4}{3}(1-x^3) & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Determine the values of the following probabilities (Hint: you should be evaluating a definite integral):

(a) $\Pr(X < \frac{1}{2})$,

(b) $\Pr(\frac{1}{4} < X < \frac{3}{4})$,

(c) $\Pr(X > \frac{1}{3})$.

Answer.

(a) $\Pr\left(X < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{4}{3}(1 - x^3)\,dx = \left[\frac{4}{3}\left(x - \frac{x^4}{4}\right)\right]_0^{\frac{1}{2}} = \frac{4}{3}\left(\frac{1}{2} - \frac{1}{4 \cdot 2^4}\right) = \frac{31}{48},$

(b) $\Pr\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{4}{3}(1 - x^3)\,dx = \left[\frac{4}{3}\left(x - \frac{x^4}{4}\right)\right]_{\frac{1}{4}}^{\frac{3}{4}} = \frac{229}{256} - \frac{85}{256} = \frac{9}{16},$

(c) $\Pr\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^{1} \frac{4}{3}(1 - x^3)\,dx = \left[\frac{4}{3}\left(x - \frac{x^4}{4}\right)\right]_{\frac{1}{3}}^{1} = 1 - \frac{107}{243} = \frac{136}{243}.$

8. Suppose Bear University and Lion University play a sequence of basketball games against each other, and the first team to win four games wins the series. Let p be the probability that Bear University wins an individual game, and assume that the games are independent. Answer the following questions

(a) What is the probability that Bear University wins the series?

(b) Does the answer to (a) depend on whether the teams always play 7 games (and the winner is the team that one the majority of games), or the teams stop playing more games as soon as one team has won 4 games (as is actually the case in practice: once the series is decided, the two teams do not keep playing more games)?

Answer.

(a) Note that Bear University could win the series in 4 different cases:

  I. Win the first 4 games.
  II. Win 3 in the first 4 games, then win the 5th game.
  III. Win 3 in the first 5 games, then win the 6th game.
  IV. Win 3 in the first 6 games, then win the 7th game.

The probability that Bear University wins the series is the sum of the probabilities of the four cases above (since they are disjoint).

$$\begin{aligned}\Pr(\text{Bear Wins}) &= \Pr(\text{Bear Wins in case I}) + \Pr(\text{Bear Wins in case II}) \\ &\quad + \Pr(\text{Bear Wins in case III}) + \Pr(\text{Bear Wins in case IV})\end{aligned}$$

$$
\begin{aligned}
&= \binom{4}{4} \times p^4 \\
&\quad + \underbrace{\binom{4}{3} \times p^3 \times (1-p)}_{\text{win 3 and lose 1 in any of first 4}} \times \underbrace{p}_{\text{win 5th}} \\
&\quad + \binom{5}{3} \times p^3 \times (1-p)^2 \times p \\
&\quad + \binom{6}{3} \times p^3 \times (1-p)^3 \times p \\
&= p^4 + \binom{4}{3} p^4 (1-p) + \binom{5}{3} p^4 (1-p)^2 + \binom{6}{3} p^4 (1-p)^3.
\end{aligned}
$$

(b) It's the same. It doesn't matter if the games are actually played. We can still consider the result of the non-actual games.

9. Does social environment affect the political development of preadults? Some work[1] shows that there exists a transmission of political values from parents to children, as manifested in their views during late adolescence. We will use this idea to solve the following questions.

Take that certain political attitudes could be shared by a parent or not. Specifically, if the parent holds the political attitude, there is $3/5$ probability that the child will also share the same attitude. On the other hand, if the parent does not have the political attitude, the child will not have it either. We exclude the possibility of any transmission of political values among children. Assume that a parent, who has probability $1/3$ of having the political attitude, has 2 children (assume that to be a son and a daughter), and answer the following questions:

(a) Conditional on the parent not holding the political attitude, is whether the son has the political attitude independent of whether the daughter has the political attitude?

(b) Conditional on the parent holding the political attitude, is whether the son has the political attitude independent of whether the daughter has the political attitude?

(c) Is whether the son has the political attitude independent of whether the daughter has the political attitude?

(d) What's the probability that neither of the 2 children have the political attitude?

(e) If the son does not have the political attitude, what's the probability that the parent has the political attitude?

[1] Jennings, M.K., and Niemi, R.G. (1968). "The transmission of political values from parents to child", *American Political Science Review* 62(1), 169-184.

(f) Bayes theorem can also be used for three events as shown below. Explain each equation below in your own words.

$$\begin{aligned}
\Pr(X \mid Y \cap Z) &= \frac{\Pr(X \cap Y \cap Z)}{\Pr(Y \cap Z)} && (1)\\
&= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z)} && (2)\\
&= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z \mid X)\Pr(X) + \Pr(Y \cap Z \mid X^c)\Pr(X^c)} && (3)
\end{aligned}$$

(g) We found that the daughter does not have the political attitude either. Using the equations in part (f), "update" your belief about whether the parent has the attitude.

Answer. For this problem, we will use the following notation:

- M: The parent holds the political attitude.
- S: The son holds the political attitude.
- D: The daughter holds the political attitude.

(a) Yes, they are independent since the probability for either child to have the political attitude is 0, given that the parent does not hold the political attitude:

$$\Pr(S \mid M^c)\Pr(D \mid M^c) = \Pr(S \cap D \mid M^c) = 0.$$

(b) Yes, they are independent since we assume there is no transmission of political values among children. So if the parent holds the political attitude, the probability for the son to also hold the political attitude is $3/5$, same for the daughter. Knowing whether the son holds the political attitude does not tell us any information about whether the daughter holds the political attitude, given that we know the parent holds the political attitude:

$$\Pr(S \mid M) = \Pr(S \mid M \cap D) = \Pr(S \mid M \cap D^c) = \frac{3}{5}.$$

(c) No, they are not independent. For example, if we know that the son holds the political attitude, we also know that the parent must also hold the political attitude, which gives us more information about whether the daughter holds the political attitude.

Conditional on the son holding the political attitude, the probability that the daughter also holds the political attitude is:

$$\Pr(D \mid S) = \Pr(D \mid S \cap M) \times \Pr(M \mid S) + \Pr(D \mid S \cap M^c) \times \Pr(M^c \mid S)$$

$$= \Pr(D \mid S \cap M) \times \Pr(M \mid S) + \Pr(D \mid S \cap M^c) \times \frac{\underbrace{\Pr(S \mid M^c)}_{=0} \Pr(M^c)}{\Pr(S)}$$
$$= \Pr(D \mid S \cap M) \times \Pr(M \mid S)$$
$$= \Pr(D \mid M) \times \underbrace{\Pr(M \mid S)}_{=1}$$
$$= \Pr(D \mid M)$$
$$= \frac{3}{5}$$

However, if we don't know whether the son holds the political attitude or not, the probability that the daughter holds the political attitude is:

$$\begin{aligned}\Pr(D) &= \Pr(D \mid M) \times \Pr(M) + \underbrace{\Pr(D \mid M^c)}_{=0} \times \Pr(M^c) \\ &= \Pr(D \mid M) \times \Pr(M) \\ &= \frac{3}{5} \times \frac{1}{3} \\ &= \frac{1}{5} \neq \Pr(D \mid S)\end{aligned}$$

(d) We need to calculate $\Pr(S^c \cap D^c)$, which can be solved by using the law of total probability.

$$\begin{aligned}\Pr(S^c \cap D^c) &= \Pr(S^c \cap D^c \mid M) \times \Pr(M) + \Pr(S^c \cap D^c \mid M^c) \times \Pr(M^c) \\ &= \Pr(S^c \mid M) \times \Pr(D^c \mid M) \times \Pr(M) + \Pr(S^c \mid M^c) \times \Pr(D^c \mid M^c) \times \Pr(M^c)\end{aligned}$$

(since S and D are independent given M, as shown in (c))

$$= \frac{2}{5} \times \frac{2}{5} \times \frac{1}{3} + 1 \times 1 \times \frac{2}{3}$$
$$= \frac{18}{25}.$$

(e) We need to calculate $\Pr(M \mid S^c)$. Using Bayes' rule, we have:

$$\begin{aligned}\Pr(M \mid S^c) &= \frac{\Pr(S^c \mid M)\Pr(M)}{\Pr(S^c)} \\ &= \frac{\Pr(S^c \mid M)\Pr(M)}{\Pr(S^c \mid M)\Pr(M) + \Pr(S^c \mid M^c)\Pr(M^c)} \\ &= \frac{\frac{2}{5} \times \frac{1}{3}}{\frac{2}{5} \times \frac{1}{3} + 1 \times \frac{2}{3}}\end{aligned}$$

$$= \frac{1}{6}.$$

(f) $$\begin{aligned}\Pr(X \mid Y \cap Z) &= \frac{\Pr(X \cap Y \cap Z)}{\Pr(Y \cap Z)} \quad \text{(Defn. of conditional probability)}\\ &= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z)} \quad \text{(Bayes' rule)}\\ &= \frac{\Pr(Y \cap Z \mid X)\Pr(X)}{\Pr(Y \cap Z \mid X)\Pr(X) + \Pr(Y \cap Z \mid X^c)\Pr(X^c)} \quad \text{(Law of total probability).}\end{aligned}$$

(g) We need to calculate $\Pr(M \mid S^c \cap D^c)$. Using the formula above, we have:

$$\begin{aligned}\Pr(M \mid S^c \cap D^c) &= \frac{\Pr(D^c \cap S^c \mid M)\Pr(M)}{\Pr(D^c \cap S^c \mid M)\Pr(M) + \Pr(D^c \cap S^c \mid M^c)\Pr(M^c)}\\ &= \frac{\frac{2}{5} \times \frac{2}{5} \times \frac{1}{3}}{\frac{3}{5} \times \frac{3}{5} \times \frac{1}{3} + 1 \times 1 \times \frac{2}{3}}\\ &= \frac{2}{27}\end{aligned}$$