Math Camp 2025 – Problem Set 8

Read the following problems carefully and justify all your work. Avoid using calculators or computers.

Partial derivatives.

1. Standard regression models often look something like this:

$$y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2$$

   (a) Find the partial derivatives of y with respect to x_1 and x_2.

   (b) Interpret both.

2. Find the first-order partial derivatives with respect to x, y, and z of $f(x, y, z) = xy^2 + yz^2$.

3. Find both second-order partial derivatives of $f(x, y) = x^2y^2$.

4. Find the second-order partial derivative of $f(x, y) = \dfrac{x}{y} + e^{xy}$.

5. Find the first- and second-order partial derivatives of $f(x, y) = \log(x + \sqrt{y})$.

6. Find the first- and second-order partial derivatives of $f(x, y) = \dfrac{x^2 + y^2}{x^3 - 4xy - y^2}$.

7. Find the second-order partial derivatives of $f(x, y) = (2x + 3y)(e^{3x} + e^{2y})$.

8. Find the second-order partial derivatives of $f(x, y, z) = x^y \log(z) - y^3x^2z + 2yz - x + 1$.

9. Find the gradient vector and Hessian matrix for the following functions:

   (a) $f(x, y) = x \log(y)$,

   (b) $f(x, y) = 3x + 4y^3$,

   (c) $f(x, y, z) = xy^2 + yz^2$,

   (d) $f(x, y) = \dfrac{3}{2}x^2 - 2xy - 5x + 2y^2 - 2y$.

Answer.

1. $y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2$

   (a) $\dfrac{\partial y}{\partial x_1} = \beta_1, \qquad \dfrac{\partial y}{\partial x_2} = \beta_2 + 2\beta_3 x_2.$

   (b) Interpretation:

- $\frac{\partial y}{\partial x_1} = \beta_1$: the effect of increasing x_1 by one unit on y keeping everything else constant is β_1.
- $\frac{\partial y}{\partial x_2} = \beta_2 + 2\beta_3 x_2$: the effect of increasing x_2 by one unit on y keeping everything else constant is $\beta_2 + 2\beta_3 x_2$. Notice that it depends on x_2.

2. $f(x, y, z) = xy^2 + yz^2$, so

$$\frac{\partial f}{\partial x} = y^2, \quad \frac{\partial f}{\partial y} = 2xy + z^2, \quad \frac{\partial f}{\partial z} = 2yz.$$

3. $f(x, y) = x^2 y^2$, so

$$\frac{\partial f}{\partial x} = 2xy^2, \quad \frac{\partial f}{\partial y} = 2x^2 y, \quad \frac{\partial^2 f}{\partial^2 x} = 2y^2, \quad \frac{\partial^2 f}{\partial^2 y} = 2x^2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4xy.$$

4. $f(x, y) = \frac{x}{y} + e^{xy}$, so

$$\frac{\partial f}{\partial x} = \frac{1}{y} + ye^{xy}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} + xe^{xy},$$

$$\frac{\partial^2 f}{\partial^2 x} = y^2 e^{xy}, \quad \frac{\partial^2 f}{\partial^2 y} = \frac{2x}{y^3} + x^2 e^{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{y^2} + (1 + xy)e^{xy}.$$

5. $f(x, y) = \log(x + \sqrt{y})$, so, assuming $x + \sqrt{y} > 0$ and $y > 0$ we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{x + \sqrt{y}}, \\
\frac{\partial f}{\partial y} &= \frac{1}{2\sqrt{y}(x + \sqrt{y})}, \\
\frac{\partial^2 f}{\partial^2 x} &= -\frac{1}{(x + \sqrt{y})^2}, \\
\frac{\partial^2 f}{\partial x \partial y} &= -\frac{1}{2\sqrt{y}(x + \sqrt{y})^2}, \\
\frac{\partial^2 f}{\partial^2 y} &= -\frac{1}{4y^{3/2}(x + \sqrt{y})} - \frac{1}{4y(x + \sqrt{y})^2}.
\end{aligned}$$

6. Given $f(x, y) = \frac{x^2 + y^2}{x^3 - 4xy - y^2}$ we have

$$\frac{\partial f}{\partial x} = \frac{-x^4 - 3x^2 y^2 - 4x^2 y - 2xy^2 + 4y^3}{(x^3 - 4xy - y^2)^2},$$

$$\frac{\partial f}{\partial y} = \frac{2x^3y + 4x^3 + 2x^2y - 4xy^2}{(x^3 - 4xy - y^2)^2},$$
$$\frac{\partial^2 f}{\partial^2 x} = \frac{-2x^6 - 12x^4y^2 - 24x^4y - 14x^3y^2 + 24x^2y^3 - 6xy^4 - 34y^4}{(x^3 - 4xy - y^2)^3},$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{6x^5y + 12x^5 + 8x^4y - 12x^3y^2 + 16x^3y + 6x^2y^3 + 12x^2y^2 + 20xy^3 - 4y^4}{(x^3 - 4xy - y^2)^3},$$
$$\frac{\partial^2 f}{\partial^2 y} = \frac{-2x^6 - 2x^5 - 32x^4 - 6x^3y^2 - 24x^3y - 6x^2y^2 + 8xy^3}{(x^3 - 4xy - y^2)^3}.$$

7. $f(x, y) = (2x + 3y)(e^{3x} + e^{2y})$:

$$\frac{\partial f}{\partial x} = 2(e^{3x} + e^{2y}) + (2x + 3y)3e^{3x}, \quad \frac{\partial f}{\partial y} = 3(e^{3x} + e^{2y}) + (2x + 3y)2e^{2y},$$

$$\frac{\partial^2 f}{\partial^2 x} = (12 + 18x + 27y)e^{3x}, \quad \frac{\partial^2 f}{\partial^2 y} = (12 + 8x + 12y)e^{2y}, \quad \frac{\partial^2 f}{\partial x \partial y} = 9e^{3x} + 4e^{2y}.$$

8. $f(x, y, z) = x^y \log z - y^3x^2z + 2yz - x + 1$:

$$\frac{\partial f}{\partial x} = yx^{y-1} \log z - 2y^3xz - 1, \quad \frac{\partial f}{\partial y} = x^y \log x \ \log z - 3y^2x^2z + 2z, \quad \frac{\partial f}{\partial z} = \frac{x^y}{z} - y^3x^2 + 2y,$$

$$\frac{\partial^2 f}{\partial^2 x} = y(y-1)x^{y-2} \log z - 2y^3z, \quad \frac{\partial^2 f}{\partial^2 y} = (\log x)^2 x^y \log z - 6yx^2z, \quad \frac{\partial^2 f}{\partial^2 z} = -\frac{x^y}{z^2},$$

$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y \log x) \log z - 6y^2xz, \quad \frac{\partial^2 f}{\partial x \partial z} = \frac{yx^{y-1}}{z} - 2y^3x, \quad \frac{\partial^2 f}{\partial y \partial z} = \frac{x^y \log x}{z} - 3y^2x^2 + 2.$$

9. Gradients ∇f and Hessians H:

   (a) $f(x, y) = x \log y$:
   $$\nabla f = \begin{pmatrix} \log y \\ x/y \end{pmatrix}, \quad \nabla^2 f = \begin{pmatrix} 0 & 1/y \\ 1/y & -x/y^2 \end{pmatrix}.$$

   (b) $f(x, y) = 3x + 4y^3$:
   $$\nabla f = \begin{pmatrix} 3 \\ 12y^2 \end{pmatrix}, \quad \nabla^2 f = \begin{pmatrix} 0 & 0 \\ 0 & 24y \end{pmatrix}.$$

   (c) $f(x, y, z) = xy^2 + yz^2$:
   $$\nabla f = \begin{pmatrix} y^2 \\ 2xy + z^2 \\ 2yz \end{pmatrix}, \quad \nabla^2 f = \begin{pmatrix} 0 & 2y & 0 \\ 2y & 2x & 2z \\ 0 & 2z & 2y \end{pmatrix}.$$

(d) $f(x,y) = \frac{3}{2}x^2 - 2xy - 5x + 2y^2 - 2y$:

$$\nabla f = \begin{pmatrix} 3x - 2y - 5 \\ -2x + 4y - 2 \end{pmatrix}, \quad \nabla^2 f = \begin{pmatrix} 3 & -2 \\ -2 & 4 \end{pmatrix}.$$

Multiple integrals. Calculate:

1. $\iint_{[0,1]^2} (xy - x^2 - y^2)\, dxdy$.
2. $\iint_{[1,2]^2} (x^2 + y^2)\, dxdy$.
3. $\iint_D (x + y)\, dxdy$, where $D = \{(x,y) \in \mathbb{R} : 0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 2x\}$.

Answer.

1. $$\begin{aligned}\iint_{[0,1]^2} (xy - x^2 - y^2)\, dxdy &= \int_0^1 \int_0^1 (xy - x^2 - y^2)\, dxdy \\ &= \int_0^1 \left[\frac{1}{2}x^2y - \frac{1}{3}x^3 - xy^2\right]_{x=0}^1 dy \\ &= \int_0^1 \left(\frac{y}{2} - \frac{1}{3} - y^2\right) dy \\ &= \left[\frac{1}{4}y^2 - \frac{1}{3}y - \frac{1}{3}y^3\right]_0^1 \\ &= -\frac{5}{12}.\end{aligned}$$

2. $$\begin{aligned}\iint_{[1,2]^2} (x^2 + y^2)\, dxdy &= \int_1^2 \int_1^2 (x^2 + y^2)\, dxdy \\ &= \int_1^2 \left[\frac{1}{3}x^3 + xy^2\right]_{x=1}^2 dy \\ &= \int_1^2 \left(\frac{7}{3} + y^2\right) dy \\ &= \left[\frac{7}{3}y + \frac{1}{3}y^3\right]_1^2 \\ &= \frac{14}{3}.\end{aligned}$$

3. $$\iint_D (x+y)\,dxdy = \int_0^1 \int_0^{2x} (x+y)\,dydx$$
$$= \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{2x} dx$$
$$= \int_0^1 \left(2x^2 + 2x^2\right) dx$$
$$= \left[\frac{4}{3}x^3\right]_0^1$$
$$= \frac{4}{3}.$$