Math Camp 2025 – Problem Set 7

Read the following problems carefully and justify all your work. Avoid using calculators or computers.

Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{E} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 8 \end{pmatrix}$$

1. For each of them, identify whether the matrix is: square, symmetric, triangular, identity, **0**, or none of the above.

2. Calculate $\operatorname{tr}(\boldsymbol{A})$.

3. Calculate $5(\operatorname{tr}(\boldsymbol{B}) + \operatorname{tr}(\boldsymbol{E}))$.

4. Calculate $\det(\boldsymbol{C}), \det(\boldsymbol{D}), \det(\boldsymbol{E})$.

Answer.

1. $\boldsymbol{A}$: square, symmetric, identity.
   $\boldsymbol{B}$: square.
   $\boldsymbol{C}$: square.
   $\boldsymbol{D}$: square, symmetric.
   $\boldsymbol{E}$: square, triangular.

2. $\operatorname{tr}(\boldsymbol{A}) = 1 + 1 = 2$.

3. $\operatorname{tr}(\boldsymbol{B}) = 1 + 5 + 9 = 15$, $\operatorname{tr}(\boldsymbol{E}) = 1 + 3 + 8 = 12$, so $5(15 + 12) = 5 \cdot 27 = 135$.

4. $\det(\boldsymbol{C}) = 2 \cdot 7 - 3 \cdot 5 = -1$, $\det(\boldsymbol{D}) = 1 \cdot 1 - 1 \cdot 1 = 0$, $\det(\boldsymbol{E}) = 1 \cdot 3 \cdot 8 = 24$.

Now consider the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 5 \\ 1 & -2 & -1 \\ 5 & -1 & 2 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{D} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \qquad \boldsymbol{E} = \begin{pmatrix} 0 & 1 & 2 \\ 5 & 1 & -1 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Is $\boldsymbol{E}^\top \boldsymbol{E}$ square? symmetric? triangular?

2. Find the trace of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$.

Answer.

1. $\boldsymbol{E}^\top \boldsymbol{E}$ is a 3×3 matrix, so square. It is symmetric because

$$(\boldsymbol{E}^\top \boldsymbol{E})^\top = \boldsymbol{E}^\top (\boldsymbol{E}^\top)^\top = \boldsymbol{E}^\top \boldsymbol{E}.$$

   We have

$$\boldsymbol{E}^\top \boldsymbol{E} = \begin{pmatrix} 30 & 14 & -5 \\ 14 & 19 & 1 \\ -5 & 1 & 5 \end{pmatrix},$$

   which is clearly not triangular.

2. $\mathrm{tr}(\boldsymbol{A}) = 0 + (-2) + 2 = 0$, $\mathrm{tr}(\boldsymbol{B}) = 4 + 3 = 7$, $\mathrm{tr}(\boldsymbol{C}) = 1 + 1 + 1 = 3$, $\mathrm{tr}(\boldsymbol{D}) = 1 + (-2) = -1$.

Invert the following matrices or give a reason why you cannot:

1. $\begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$
2. $\begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}$
3. $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$
4. $\begin{pmatrix} 11 & 3 & 5 \\ 3 & 2 & 19 \\ 0 & 0 & 0 \end{pmatrix}$
5. $\begin{pmatrix} 3 & 8 & 6 \\ 0 & -3 & -5 \\ -9 & 0 & 4 \end{pmatrix}$

Answer.

1. Invertible, det is 1, and we can calculate the inverse $\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$ using the formula we saw in lecture.

2. The det is 0, so it's not invertible.

3. The easiest way is to calculate the determinant and the adjoint:

$$\det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 2 \cdot (2 \cdot 2 - 1 \cdot 1) - 1 \cdot (1 \cdot 2 - 1 \cdot 0) + 0 \cdot (1 \cdot 1 - 2 \cdot 0) = 6 - 2 = 4,$$

and

$$\operatorname{adj}\begin{pmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix} = \begin{pmatrix}3 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 3\end{pmatrix}^{\mathsf{T}} = \begin{pmatrix}3 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 3\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix}3 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 3\end{pmatrix}.$$

We can also use Gaussian elimination:

$$\left(\begin{array}{ccc|ccc}2 & 1 & 0 & 1 & 0 & 0\\ 1 & 2 & 1 & 0 & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 1 & 2 & 1 & 0 & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 1 & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0\\ 0 & 1 & 2 & 0 & 0 & 1\end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0\\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & 1\end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0\\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4}\end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4}\end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4}\\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2}\\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & \frac{3}{4}\end{array}\right).$$

and we get to the same answer.

4. We have

$$\det\begin{pmatrix}11 & 3 & 5\\ 3 & 2 & 19\\ 0 & 0 & 0\end{pmatrix} = 11\cdot(2\cdot 0 - 19\cdot 0) - 3\cdot(3\cdot 0 - 19\cdot 0) + 5\cdot(3\cdot 0 - 2\cdot 0) = 0 - 0 + 0 = 0,$$

so the determinant is 0, and therefore the matrix is not invertible.

5. We have

$$\det\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 4\\ 5 & 6 & 0\end{pmatrix} = 1\cdot(-24) - 0\cdot 18 + 5\cdot 5 = 1,$$

and

$$\operatorname{adj}\begin{pmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{pmatrix} = \begin{pmatrix}-24 & 20 & -5\\18 & -15 & 4\\5 & -4 & 1\end{pmatrix}^{\top} = \begin{pmatrix}-24 & 18 & 5\\20 & -15 & -4\\-5 & 4 & 1\end{pmatrix}.$$

Therefore,

$$\begin{pmatrix}1 & 2 & 3\\0 & 1 & 4\\5 & 6 & 0\end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix}-24 & 18 & 5\\20 & -15 & -4\\-5 & 4 & 1\end{pmatrix} = \begin{pmatrix}-24 & 18 & 5\\20 & -15 & -4\\-5 & 4 & 1\end{pmatrix}.$$

Find all the solutions to the following systems of linear equations:

1. $\begin{cases}-3x + 5y + 5z = -43\\ \quad x - 4y - 2z = 31\\ \qquad 3x - 4z = 7\end{cases}$

2. $\begin{cases}x - 2y - z = -15\\ -x - y + z = -6\\ x - 6y - z = -43\end{cases}$

Answer.

1. We can use Gaussian elimination:

$$\left(\begin{array}{ccc|c}-3 & 5 & 5 & -43\\1 & -4 & -2 & 31\\3 & 0 & -4 & 7\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1 & -4 & -2 & 31\\-3 & 5 & 5 & -43\\3 & 0 & -4 & 7\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1 & -4 & -2 & 31\\0 & -7 & -1 & 50\\0 & 12 & 2 & -86\end{array}\right)$$
$$\Rightarrow \left(\begin{array}{ccc|c}1 & -4 & -2 & 31\\0 & 1 & \frac{1}{7} & -\frac{50}{7}\\0 & 12 & 2 & -86\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1 & -4 & -2 & 31\\0 & 1 & \frac{1}{7} & -\frac{50}{7}\\0 & 0 & \frac{2}{7} & -\frac{2}{7}\end{array}\right),$$

so the solution is $x = 1$, $y = -7$, $z = -1$.

2. We use Gaussian elimination and we obtain

$$\left(\begin{array}{ccc|c}1 & -2 & -1 & -15\\-1 & -1 & 1 & -6\\1 & -6 & -1 & -43\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -15\\0 & -3 & 0 & -21\\0 & -4 & 0 & -28\end{array}\right) \Rightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -15\\0 & 1 & 0 & 7\\0 & -4 & 0 & -28\end{array}\right)$$
$$\Rightarrow \left(\begin{array}{ccc|c}1 & -2 & -1 & -15\\0 & 1 & 0 & 7\\0 & 0 & 0 & 0\end{array}\right).$$

So $y = 7$, and x, z need to satisfy $x - 14 - z = -15$, i.e., $x = z - 1$; we can take any $z \in \mathbb{R}$.