Math Camp 2025 – Problem Set 6

Read the following problems carefully and justify all your work. Avoid using calculators or computers.

1. Matrix Arithmetic. Consider the following vectors and matrices:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \boldsymbol{C} = \begin{pmatrix} 3 & 2 & -4 \\ -8 & 0 & 6 \end{pmatrix} \quad \boldsymbol{D} = \begin{pmatrix} 6 & -2 \\ -1 & 3 \\ -3 & 8 \end{pmatrix} \quad \boldsymbol{a} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

Complete the following operations or give a reason why you cannot:

1. $3\boldsymbol{a} - 2\boldsymbol{b}$
2. $\|\boldsymbol{b}\|$
3. $\|\boldsymbol{a} - \boldsymbol{b}\|$
4. $\|\boldsymbol{C}\boldsymbol{a}\|$
5. $\boldsymbol{C}\boldsymbol{D}$
6. $\boldsymbol{D}\boldsymbol{C}$
7. $\boldsymbol{a} \cdot \boldsymbol{b}$
8. $\boldsymbol{a}\boldsymbol{b}^\top$
9. $\boldsymbol{a}\boldsymbol{b}^\top\boldsymbol{B} - \boldsymbol{D}\boldsymbol{C}$
10. $\boldsymbol{b}\boldsymbol{D}$
11. $\boldsymbol{A}^\top\boldsymbol{A}$
12. $\boldsymbol{b}^\top\boldsymbol{D}$
13. $\boldsymbol{B}^2$

Answer.

1. $3\boldsymbol{a} - 2\boldsymbol{b} = 3\begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \\ 12 \end{pmatrix} - \begin{pmatrix} 6 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -9 \\ 13 \\ 10 \end{pmatrix}$

2. $\|\boldsymbol{b}\| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$

3. $\|\boldsymbol{a} - \boldsymbol{b}\| = \left\| \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} \right\| = \sqrt{(-4)^2 + 5^2 + 3^2} = \sqrt{16 + 25 + 9} = \sqrt{50}$

4. We have

$$\boldsymbol{C}\boldsymbol{a} = \begin{pmatrix} 3 & 2 & -4 \\ -8 & 0 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} 3 \cdot (-1) + 2 \cdot 3 + (-4) \cdot 4 \\ -8 \cdot (-1) + 0 \cdot 3 + 6 \cdot 4 \end{pmatrix} \\
&= \begin{pmatrix} -3 + 6 - 16 \\ 8 + 0 + 24 \end{pmatrix} = \begin{pmatrix} -13 \\ 32 \end{pmatrix},
\end{aligned}
$$

so

$$
\|\boldsymbol{Ca}\| = \sqrt{(-13)^2 + 32^2} = \sqrt{169 + 1024} = \sqrt{1193}.
$$

5. We have

$$
\begin{aligned}
\boldsymbol{CD} &= \begin{pmatrix} 3 & 2 & -4 \\ -8 & 0 & 6 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -1 & 3 \\ -3 & 8 \end{pmatrix} \\
&= \begin{pmatrix} 3 \cdot 6 + 2 \cdot (-1) + (-4) \cdot (-3) & 3 \cdot (-2) + 2 \cdot 3 + (-4) \cdot 8 \\ -8 \cdot 6 + 0 \cdot (-1) + 6 \cdot (-3) & -8 \cdot (-2) + 0 \cdot 3 + 6 \cdot 8 \end{pmatrix} \\
&= \begin{pmatrix} 28 & -32 \\ -66 & 64 \end{pmatrix}.
\end{aligned}
$$

6. We have

$$
\begin{aligned}
\boldsymbol{DC} &= \begin{pmatrix} 6 & -2 \\ -1 & 3 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} 3 & 2 & -4 \\ -8 & 0 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 6 \cdot 3 + (-2) \cdot (-8) & 6 \cdot 2 + (-2) \cdot 0 & 6 \cdot (-4) + (-2) \cdot 6 \\ -1 \cdot 3 + 3 \cdot (-8) & -1 \cdot 2 + 3 \cdot 0 & -1 \cdot (-4) + 3 \cdot 6 \\ -3 \cdot 3 + 8 \cdot (-8) & -3 \cdot 2 + 8 \cdot 0 & -3 \cdot (-4) + 8 \cdot 6 \end{pmatrix} \\
&= \begin{pmatrix} 34 & 12 & -36 \\ -27 & -2 & 22 \\ -73 & -6 & 60 \end{pmatrix}.
\end{aligned}
$$

7. $\boldsymbol{a} \cdot \boldsymbol{b} = -1 \cdot 3 + 3 \cdot (-2) + 4 \cdot 1 = -3 - 6 + 4 = -5.$

8. We have

$$
\boldsymbol{ab}^\top = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \end{pmatrix}
$$

$$= \begin{pmatrix} -1 \cdot 3 & -1 \cdot (-2) & -1 \cdot 1 \\ 3 \cdot 3 & 3 \cdot (-2) & 3 \cdot 1 \\ 4 \cdot 3 & 4 \cdot (-2) & 4 \cdot 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 & -1 \\ 9 & -6 & 3 \\ 12 & -8 & 4 \end{pmatrix}.$$

9. We have

$$\begin{aligned} \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{B} &= \begin{pmatrix} -3 & 2 & -1 \\ 9 & -6 & 3 \\ 12 & -8 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 & -4 \\ 3 & -6 & 12 \\ 4 & -8 & 16 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} \boldsymbol{a}\boldsymbol{b}^\top \boldsymbol{B} - \boldsymbol{D}\boldsymbol{C} &= \begin{pmatrix} -1 & 2 & -4 \\ 3 & -6 & 12 \\ 4 & -8 & 16 \end{pmatrix} - \begin{pmatrix} 34 & 12 & -36 \\ -27 & -2 & 22 \\ -73 & -6 & 60 \end{pmatrix} \\ &= \begin{pmatrix} -35 & -10 & 32 \\ 30 & -4 & -10 \\ 77 & -2 & -44 \end{pmatrix}. \end{aligned}$$

10. We have

$$\begin{aligned} \boldsymbol{b}\boldsymbol{D} &= \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -1 & 3 \\ -3 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 3 \cdot 6 + (-2) \cdot (-1) + 1 \cdot (-3) & 3 \cdot (-2) + (-2) \cdot 3 + 1 \cdot 8 \end{pmatrix} \\ &= \begin{pmatrix} 18 + 2 - 3 & -6 - 6 + 8 \end{pmatrix} = \begin{pmatrix} 17 & -4 \end{pmatrix}. \end{aligned}$$

11. We have

$$\begin{aligned} \boldsymbol{A}^\top \boldsymbol{A} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 & 1 \cdot 2 + 2 \cdot 1 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 2 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}. \end{aligned}$$

12. We have

$$\begin{aligned}
\boldsymbol{b}^\top \boldsymbol{D} &= \begin{pmatrix} 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -1 & 3 \\ -3 & 8 \end{pmatrix} \\
&= \begin{pmatrix} 3 \cdot 6 + (-2) \cdot (-1) + 1 \cdot (-3) & 3 \cdot (-2) + (-2) \cdot 3 + 1 \cdot 8 \end{pmatrix} \\
&= \begin{pmatrix} 18 + 2 - 3 & -6 - 6 + 8 \end{pmatrix} = \begin{pmatrix} 17 & -4 \end{pmatrix}.
\end{aligned}$$

13. We have

$$\begin{aligned}
\boldsymbol{B}^2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 \\ 1 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 & 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.
\end{aligned}$$

Optional. What is a general formula for $\boldsymbol{B}^n = \underbrace{\boldsymbol{B} \cdot \ldots \cdot \boldsymbol{B}}_{n \text{ times}}$?

Answer. Notice that if $\boldsymbol{X} \in \mathbb{R}^{3 \times 3}$ then $\boldsymbol{XB}$ is: the third column of $\boldsymbol{X}$, then the second one, and then the first column plus the third one. So, in $\boldsymbol{B}^n$ the second column will stay the same, i.e., it will be $(0, 1, 0)$. And we will have zeros in the second row in the first and third columns:

$$\boldsymbol{B}^n = \begin{pmatrix} ? & 0 & ? \\ 0 & 1 & 0 \\ ? & 0 & ? \end{pmatrix}.$$

The ? cells start as $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and then become $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$, etc. These are Fibonacci numbers: $F_1 = 0$, $F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$. Thus, we have:

$$\boldsymbol{B}^n = \begin{pmatrix} F_n & 0 & F_{n+1} \\ 0 & 1 & 0 \\ F_{n+1} & 0 & F_{n+2} \end{pmatrix}.$$