Math Camp 2025 – Problem Set 5

Read the following problems carefully and justify all your work. Avoid using calculators or computers.

1. Indefinite Integrals. Find the following indefinite integrals.

1. $\int (3x^3 + 2x^2 - e^x)\, dx$
2. $\int \frac{2x}{x^2}\, dx$
3. $\int \frac{1}{x^2}\, dx$
4. $\int 2x(x^2 - 64)^2\, dx$
5. $\int \frac{1}{x \log(x)}\, dx$
6. $\int (\exp(5x^3)x^2 - x + 2)\, dx$
7. $\int (10 - x)^{10}\, dx$

Answer.

1. $\int (3x^3 + 2x^2 - e^x)\, dx = \frac{3}{4}x^4 + \frac{2}{3}x^3 - e^x + c.$
2. $\int \frac{2x}{x^2}\, dx = \int \frac{2}{x}\, dx = 2\log(x).$
3. $\int \frac{1}{x^2}\, dx = \frac{-1}{x}.$
4. $\int 2x(x^2 - 64)^2\, dx = \int f(g(x))g'(x)\, dx = F(g(x)) + c$ with $f(x) = x^2$, $g(x) = x^2 - 64$, $F' = f$. Now, $F = \frac{1}{3}x^3$, so $\int 2x(x^2 - 64)^2\, dx = \frac{1}{3}(x^2 - 64)^3 + c.$
5. $\int \frac{1}{x \log(x)}\, dx = \int f(g(x))g'(x)\, dx = F(g(x)) + c$ with $f(x) = 1/x$, $g(x) = \log(x)$, $F' = f$. Now, $F = \log(x)$, so $\int \frac{1}{x \log(x)}\, dx = \log(\log(x)) + c.$
6. $\int (\exp(5x^3)x^2 - x + 2)\, dx = \frac{1}{15}\exp(5x^3) - \frac{1}{2}x^2 + 2x + c.$

7. $\int (10-x)^{10}\,dx = -\frac{1}{11}(10-x)^{11}$.

2. Definite and Improper Integrals. Calculate the following integrals.

1. $\int_4^5 2x\,dx$
2. $\int_{e^{\sqrt{2}}}^{e^2} \frac{\log(x)}{x}\,dx$
3. $\int_{-\infty}^{0} e^x\,dx$
4. $\int_2^{+\infty} \frac{2x-1}{(x^2-x)^2}\,dx$
5. $\int_1^9 2y^5\,dy$
6. $\int_{-1}^{0} (3x^2-1)\,dx$
7. $\int_{-1}^{1} (14+x^2)\,dx$
8. $\int_1^{-1} (14+x^2)\,dx$
9. $\int_1^2 \frac{1}{x}\,dx$
10. $\int_1^2 \frac{1}{x^2}\,dx$

Answer.

1. $\int_4^5 2x\,dx = x^2\Big|_4^5 = 5^2 - 4^2 = 9.$
2. First, note that $\int \frac{\log(x)}{x}\,dx$ is $\int f'(g(x))g'(x)\,dx = f(g(x)) + c$ with $f(x) = \frac{1}{2}x^2$ and $g(x) = \log(x)$, so $\int \frac{\log(x)}{x}\,dx = \frac{1}{2}\log(x)^2 + c$. Hence

$$\int_{e^{\sqrt{2}}}^{e^2} \frac{\log(x)}{x}\,dx = \frac{1}{2}\log(x)^2\Big|_{e^{\sqrt{2}}}^{e^2} = \frac{1}{2}\log(e^2)^2 - \frac{1}{2}\log(e^{\sqrt{2}})^2 = \frac{1}{2}(2^2 - \sqrt{2}^2) = 1.$$

3. $\displaystyle\int_{-\infty}^{0} e^x \, dx = \lim_{a\to-\infty} e^x\big|_a^0 = \lim_{a\to-\infty}(e^0 - e^a) = 1.$

4. $\displaystyle\int \frac{2x-1}{(x^2-x)^2}\, dx = \frac{-1}{x^2-x}$, so

$$\int_{2}^{+\infty} \frac{2x-1}{(x^2-x)^2}\, dx = \lim_{b\to+\infty} \frac{-1}{x^2-x}\bigg|_2^b = \lim_{b\to+\infty}\left(\frac{-1}{b^2-b} - \frac{-1}{2^2-2}\right) = \frac{1}{2}.$$

5. $\displaystyle\int_1^9 2y^5\, dy = \frac{1}{3}y^6\bigg|_1^9 = \frac{1}{3}(9^6 - 1^6) = \frac{531440}{3}.$

6. $\displaystyle\int_{-1}^{0} (3x^2-1)\, dx = x^3 - x\big|_{-1}^0 = (-1)^3 - (-1) - (0^3 - 0) = 0.$

7. $\displaystyle\int_{-1}^{1} (14+x^2)\, dx = 14x + \frac{1}{3}x^3\bigg|_{-1}^1 = (14 + \frac{1}{3}) - (-14 - \frac{1}{3}) = \frac{86}{3}.$

8. $\displaystyle\int_{1}^{-1} (14+x^2)\, dx = -\int_{1}^{-1}(14+x^2)\, dx = -\frac{86}{3}.$

9. $\displaystyle\int_1^2 \frac{1}{x}\, dx = \log(x)\big|_1^2 = \log(2) - \log(1) = \log(2).$

10. $\displaystyle\int_1^2 \frac{1}{x^2}\, dx = \frac{-1}{x}\bigg|_1^2 = \frac{-1}{2} - \frac{-1}{1} = \frac{1}{2}.$

3. Integration by parts. Calculate the following integrals.

1. $\displaystyle\int \frac{\log(x)}{x^3}\, dx$

2. $\displaystyle\int x^2 e^x\, dx$

3. $\displaystyle\int_1^e x\log(x)\, dx$

4. $\displaystyle\int \frac{x^3}{(x^2+7)^2}\, dx$

5. $\displaystyle\int (\log(x))^2\, dx$

Answer.

1. We have

$$\int \frac{\log(x)}{x^3}\,dx = \log(x)\frac{-1}{2x^2} - \int \frac{1}{x}\frac{-1}{2x^2}\,dx$$
$$= -\frac{\log(x)}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\,dx$$
$$= -\frac{\log(x)}{2x^2} - \frac{1}{4}\frac{1}{x^2}.$$

2. We have

$$\int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$$
$$= x^2e^x - 2xe^x + \int 2e^x\,dx$$
$$= x^2e^x - 2xe^x + 2e^x + c$$
$$= (x^2 - 2x + 2)e^x + c.$$

3. We have

$$\int x\log(x)\,dx = \frac{1}{2}x^2\log(x) - \int \frac{1}{2}x^2\frac{1}{x}\,dx$$
$$= \frac{1}{2}x^2\log(x) - \frac{1}{2}\int x\,dx$$
$$= \frac{1}{2}x^2\log(x) - \frac{1}{4}x^2.$$

Hence, $\int_1^e x\log(x)\,dx = \left(\frac{1}{2}x^2\log(x) - \frac{1}{4}x^2\right)\Big|_1^e = \left(\frac{1}{2}e^2\log(e) - \frac{1}{4}e^2\right) - \left(\frac{1}{2}1\log(1) - \frac{1}{4}1\right) = \frac{1}{4}(e^2+1).$

4. We have

$$\int \frac{x^3}{(x^2+7)^2}\,dx = \int x^2\frac{x}{(x^2+7)^2}\,dx$$
$$= x^2\cdot\frac{1}{2}\frac{-1}{x^2+7} - \int 2x\cdot\frac{1}{2}\frac{-1}{x^2+7}\,dx$$
$$= -\frac{x^2}{2(x^2+7)} + \int \frac{x}{x^2+7}\,dx$$
$$= -\frac{x^2}{2(x^2+7)} + \frac{1}{2}\log(x^2+7) + c.$$

5. We have

$$\begin{aligned}
\int \log(x)^2\,dx &= \int \log(x)\log(x)\,dx \\
&= (x\log(x) - x)\log(x) - \int (x\log(x) - x)\frac{1}{x}\,dx \\
&= (x\log(x) - x)\log(x) - \int (\log(x) - 1)\,dx \\
&= (x\log(x) - x)\log(x) - (x\log(x) - x) + x + c \\
&= (x\log(x) - x)(\log(x) - 1) + x + c \\
&= x(\log(x) - 1)^2 + x + c.
\end{aligned}$$