Math Camp 2025 – Problem Set 4

Read the following problems carefully and justify all your work. Avoid using calculators or computers.

1. Extrema. Find the local maxima and minima of the following functions.

1. $f(x) = \frac{1}{3}x^3$
2. $f(x) = \dfrac{x}{e^x}$
3. $f(x) = \dfrac{x^2 - 1}{x - 1}$
4. $f(x) = x^2(x - 1)$
5. $f(x) = e^{2x} + 3e^{-4x}$
6. $f(x) = xe^{2x}$
7. $f(x) = \log\left((3x - 1)^2\right)$
8. $f(x) = \dfrac{5^x}{5}$
9. $f(x) = (1 + x^2)^3$
10. $f(x) = h(g(x))$, where $h(x) = \log(x)$ and $g(x) = x^2$

Answer.

1. f is increasing, so it doesn't have local extrema.
2. f is twice differentiable with $f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{1 - x}{e^x}$ and $f''(x) = \dfrac{-e^x - (1 - x)e^x}{e^{2x}} = \dfrac{-2 + x}{e^x}$ continuous. If x is a local extremum we must have $f'(x) = 0$, i.e., $x = 1$. Now, $f''(1) = -\frac{1}{e} < 0$ so 1 is a local maximum.
3. We have $f(x) = x + 1$ except at $x = 1$, and that's increasing, so f doesn't have local extrema.
4. f is twice differentiable with $f'(x) = 3x^2 - 2x$ and $f''(x) = 6x - 2$ continuous. If x is a local extremum we must have $f'(x) = 0$, i.e., $3x^2 - 2x = (3x - 2)x = 0$, i.e., $x = 0$ or $x = \frac{2}{3}$. Now $f''(0) = -2 < 0$, so f has a local maximum at 0, and $f''(\frac{2}{3}) = 2 > 0$, so f has a local minimum at $\frac{2}{3}$.

5. f is twice differentiable with $f'(x) = 2e^{2x} - 12e^{-4x}$ and $f''(x) = 4e^{2x} + 48e^{-4x} > 0$, so f is convex. If x is a local extremum we must have $f'(x) = 0$, i.e., $e^{2x} = 6e^{-4x}$. Taking log this is $2x = \log(6) - 4x$, or $x = \frac{1}{6}\log(6)$. Since f is convex this is the minimum.

6. f is twice differentiable with $f'(x) = (1+2x)e^{2x}$ and $f''(x) = 2e^{2x} + (1+2x)2e^{2x} = (4+4x)e^{2x}$ continuous. If x is a local extremum we must have $f'(x) = 0$, i.e., $x = -\frac{1}{2}$. We have $f''(-\frac{1}{2}) = 2e^{-1} > 0$, so $-\frac{1}{2}$ is a local minimum.

7. f is differentiable with $f'(x) = \frac{6}{3x-1}$ which is never 0, so it doesn't have local extrema.

8. f is increasing, so it doesn't have local extrema.

9. f is twice differentiable with $f'(x) = 6(1+x^2)^2x$ and $f''(x) = 24(1+x^2)x^2 + 6(1+x^2)^2 = 6(1+x^2)(4x^2+1+x^2) = 6(1+x^2)(1+5x^2) > 0$, so f is convex. Now, $f'(x) = 0$ iff $x = 0$, so it's minimized at 0.

10. $f(x) = \log(x^2)$ is differentiable with $f'(x) = \frac{2}{x}$ which is never 0, so it doesn't have local extrema.

2. Concavity. For each function above, identify the intervals on which the function is convex and those on which it is concave.

Answer.

1. $f''(x) = 2x$, so f is concave for $x \leqslant 0$ and convex for $x \geqslant 0$.

2. $f''(x) = \dfrac{-2+x}{e^x}$, so f is concave for $x \leqslant 2$ and convex for $x \geqslant 2$.

3. We have $f(x) = x + 1$ except at $x = 1$, which is linear, so f is convex and concave.

4. $f''(x) = 6x - 2$ continuous, so f is concave for $x \leqslant \frac{1}{3}$ and convex for $x \geqslant \frac{1}{3}$.

5. $f''(x) = 4e^{2x} + 48e^{-4x} > 0$, so f is convex.

6. $f''(x) = (4+4x)e^{2x}$, so f is concave for $x \leqslant -1$ and convex for $x \geqslant -1$.

7. f is differentiable with $f'(x) = \frac{6}{3x-1}$ which is never 0, so it doesn't have local extrema. $f''(x) = -\frac{18}{(3x-1)^2} < 0$ so it's concave in $(-\infty, \frac{1}{3})$ and in $(\frac{1}{3}, +\infty)$.

8. $f'(x) = 5^{x-1}\log(5)$ and $f''(x) = 5^{x-1}\log(5)^2 > 0$, so f is convex.

9. $f''(x) = 6(1+x^2)(1+5x^2) > 0$, so f is convex.

10. $f''(x) = -\frac{2}{x^2} < 0$, so f is concave in $(-\infty, 0)$ and in $(0, +\infty)$.

3. L'Hôpital's Rule. Compute the following limits.

1. $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$
2. $\lim_{x \to 0} \frac{8^x - 4^x}{x^3 - x^2 - x}$
3. $\lim_{x \to +\infty} \frac{e^{2x}}{2x + 144}$
4. $\lim_{x \to +\infty} \frac{2 + \log(x)}{x^2 + 3}$
5. $\lim_{x \to 0} (x \log(x) - x)$

Answer.

1. $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x \to 9} \frac{1}{1/(2\sqrt{x})} = 6.$
2. $\lim_{x \to 0} \frac{8^x - 4^x}{x^3 - x^2 - x} = \lim_{x \to 0} \frac{8^x \log(8) - 4^x \log(4)}{3x^3 - 2x^2 - 1} = \frac{\log(8) - \log(4)}{-1} = -\log(8/4) = -\log(2).$
3. $\lim_{x \to +\infty} \frac{e^{2x}}{2x + 144} = \lim_{x \to +\infty} \frac{2e^{2x}}{2} = +\infty.$
4. $\lim_{x \to +\infty} \frac{2 + \log(x)}{x^2 + 3} = \lim_{x \to +\infty} \frac{1/x}{2x} = 0.$
5. We saw in lecture that $\lim_{x \to 0} x \log(x) = 0$, so $\lim_{x \to 0} (x \log(x) - x) = 0$.

4. Applied Problem. Political scientists often employ rational choice theory to study politics. Political actors such as legislators are assumed to have goals, and to choose actions designed to achieve them. This is operationalized by defining an actor's *utility functions* and *feasible actions*, and determining which feasible action maximizes her utility.

For example, say a legislator i's utility function u was defined by $u_i(c) = v - c^2$, where v is the legislator's vote share in an election, and c is the portion of her wealth the legislator spent on the campaign. That is, the legislator gains utility from gaining votes, but loses utility from spending her wealth to get them. Now say vote share was determined entirely by campaign spending such that $v = c$; what level of campaign spending maximizes the legislator's utility?

Answer. The utility function $u_i(c) = c - c^2$ is concave, and $u_i'(c) = 1 - 2c = 0$ iff $c = \frac{1}{2}$, so the optimal level of campaign spending is $\frac{1}{2}$.