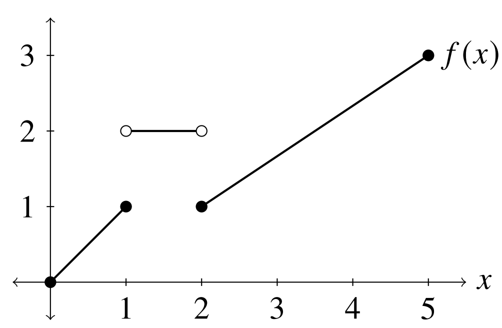
Math Camp 2025 – Problem Set 3

Read the following problems carefully and justify everything you do.

1. Limits. Find the following limits or show that they don't exist.

1. $\lim_{x\to+\infty} \frac{e}{x}$.
2. $\lim_{x\to-\infty} \frac{e}{x}$.
3. $\lim_{x\to 3} \frac{x}{x^3-27}$.
4. $\lim_{x\to 3} \frac{x-3}{x^3-27}$.
5. $\lim_{x\to\infty} \frac{x+1}{2x}$.
6. $\lim_{x\to\infty} \left(\frac{1}{2}\right)^x$.
7. $\lim_{x\to\infty} \frac{3x^3+2x^2-x+3}{4x^4+3x^3+2x^2+x+4}$.
8. $\lim_{x\to 0} \frac{1}{x^2}$.
9. (a) $\lim_{x\to 1} f(x)$.
   (b) $\lim_{x\to 2} f(x)$.
   (c) $\lim_{x\to 5} f(x)$.

   where $f : [0,5] \to \mathbb{R}$ is given by the following graph.



Answer.

1. $\lim_{x\to+\infty} \frac{e}{x} = 0$.

2. $\lim_{x\to-\infty} \frac{e}{x} = 0$.

3. $\lim_{x\to 3} \frac{x}{x^3-27} = \infty$ since $x^3 - 27 \to 0$ and $x \to 3$.

4. $\frac{x-3}{x^3-27} = \frac{x-3}{(x-3)(x^2+3x+9)} = \frac{1}{x^2+3x+9} \to \frac{1}{27}$ when $x \to 3$.

5. $\frac{x+1}{2x} = \frac{x(1+\frac{1}{x})}{2x} = \frac{1+\frac{1}{x}}{2} \to \frac{1}{2}$ when $x \to \infty$.

6. $\left(\frac{1}{2}\right)^x = \frac{1}{2^x} \to 0$ since $2^x \to \infty$ when $x \to +\infty$.

7. $\frac{3x^3+2x^2-x+3}{4x^4+3x^3+2x^2+x+4} = \frac{x^3(3+\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x^3})}{x^4(4+\frac{3}{x}+\frac{2}{x^2}+\frac{1}{x^3}+\frac{4}{x^4})} = \frac{3+\frac{2}{x}-\frac{1}{x^2}+\frac{3}{x^3}}{x(4+\frac{3}{x}+\frac{2}{x^2}+\frac{1}{x^3}+\frac{4}{x^4})} \to 0$ as $x \to \infty$.

8. $\lim_{x\to 0} \frac{1}{x^2} = +\infty$.

9. (a) $\lim_{x\to 1} f(x)$ doesn't exist because $\lim_{x\to 1^-} f(x) = 1$ but $\lim_{x\to 1^+} f(x) = 2$.
   (b) $\lim_{x\to 2} f(x)$ doesn't exist because $\lim_{x\to 2^-} f(x) = 2$ but $\lim_{x\to 2^+} f(x) = 1$.
   (c) $\lim_{x\to 5} f(x) = 3$. (Notice that f is only defined on the left of 5.)

2. Continuity.

1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x & \text{if } x \geqslant 1. \end{cases}$$

   Is it continuous?

2. The following functions are defined for all $x \in \mathbb{R}$ except for one point x_0. Find x_0 and determine if they can be defined at x_0 so that they are continuous on $\mathbb{R}$.

   (a) $f(x) = \frac{x-3}{x^3-27}$.
   (b) $f(x) = \frac{1}{x}$.
   (c) $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ x & \text{if } x > 1. \end{cases}$.

(d) $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } x > -1. \end{cases}$.

Answer.

1. Yes, it is, because $f(1) = 1$ and $\lim_{x \to 1} f(x) = 1$, the latter because $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1$.

2. (a) f is defined everywhere except when $x^3 - 27 = 0$, i.e., when $x = 3$. When $x \to 3$ we have

$$\frac{x-3}{x^3-27} = \frac{x-3}{(x-3)(x^2+3x+9)} = \frac{1}{x^2+3x+9} \to \frac{1}{27},$$

so we can define $f(3) = \frac{1}{27}$ and the result is continuous.

(b) f is defined everywhere except at 0. We have $\lim_{x \to 0} f(x) = \infty$, so we can't make f continuous by choosing $f(0)$.

(c) Undefined at 1. Look at the lateral limits—clearly by setting $f(1) = 1$ we get a continuous function.

(d) Undefined at 1. We have $\lim_{x \to -1^-} f(x) = 1$ but $\lim_{x \to -1^+} f(x) = -1$, so we can't make it continuous by choosing $f(1)$.