Math Camp 2025 – Problem Set 2

Read the following problems carefully and justify everything you do.

1. Lines and Plots.

1. Find the linear function $f(x) = ax + b$ that goes through the points $(-1, -3)$ and $(1, 1)$.

2. Say you were interested in the relationship between the amount of federal grant funds distributed by executive agencies in a jurisdiction and mean annual income. Suppose after collecting data and fitting a regression, you determined the relationship to be

$$Y = 2 + 0.5x,$$

where Y is the amount of federal grants distributed in millions and x is mean annual income in units of 1,000. Draw a graph showing this relationship for $x \in [0, 100]$ (it may be useful to use units of ten when labeling the axes). How much federal grant money is distributed to jurisdictions with a mean annual income of $25,000? $50,000? $100,000?

Answer.

1. We look for $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ satisfies $f(-1) = -3$ and $f(1) = 1$. The equations are

$$a \cdot (-1) + b = -3$$

and

$$a \cdot 1 + b = 1.$$

Subtract and get

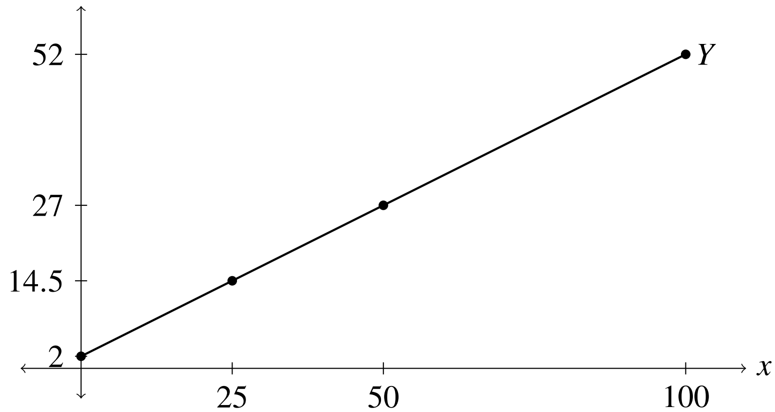
$$a \cdot (-1 - 1) = -3 - 1,$$

i.e., $-2a = -4$, so $a = 2$. Replace in the first equation and obtain

$$2 \cdot (-1) + b = -3,$$

so $b = -3 + 2 = -1$.

2. Graph:



For \$25,000 we have $x = 25$, so $Y = 2 + 0.5 \times 25 = 14.5$, and the grant money is \$14.5 million. For \$50,000 it's \$27 million, and for \$100,000 it's \$52 million.

2. Sets.

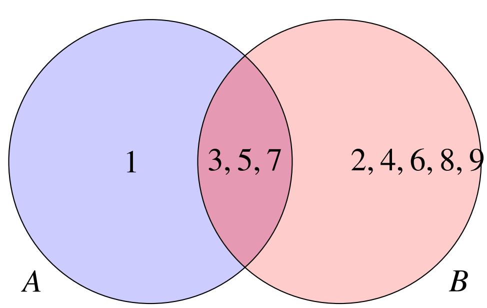
1. Let $U = \{i \in \mathbb{N} : 0 < i < 11\}$, $A = \{1, 3, 5, 7\}$, and $B = \{i \in \mathbb{N} : 1 < i < 10\}$.
   (a) Find $A \cup B$.
   (b) Find $A \cap B$.
   (c) Depict these sets in a Venn diagram.

2. For any two sets A and B, what if anything can we say about $B \setminus (B \setminus A)$?

   Comment. This can be hard. Try some examples: $B = \{1, 2, 3\}$, $A = \{1, 4\}$. Then $B = \{1, 2\}$, $A = \{3, 4\}$. Then $B = \{1, 2\}$, $A = \{2\}$. Can you spot the pattern? Can you prove it?

Answer.

1. (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
   (b) $A \cap B = \{3, 5, 7\}$.
   (c) Venn diagram:



2. For all x we have

$$\begin{aligned} x \in B \setminus (B \setminus A) \quad &\text{iff} \quad x \in B \text{ and } x \notin (B \setminus A) \\ &\text{iff} \quad x \in B \text{ and not } x \in (B \setminus A) \\ &\text{iff} \quad x \in B \text{ and not } (x \in B \text{ and } x \notin A) \\ &\text{iff} \quad x \in B \text{ and } (x \notin B \text{ or } x \in A) \\ &\text{iff} \quad (x \in B \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \in A) \\ &\text{iff} \quad \text{false or } x \in B \cap A \\ &\text{iff} \quad x \in B \cap A, \end{aligned}$$

so $B \setminus (B \setminus A) = B \cap A$.

Notice that I used properties of ‘and,’ ‘or’ and ‘not’ that may not be completely obvious.

3. Functions.

1. Factor $-7\theta^2 + 21\theta - 14$.
2. Expand and simplify $(2x - 3)(5x + 7)$.
3. Factor $q^2 - 10q + 9$.
4. Factor and reduce $\dfrac{\beta - \alpha}{\alpha^2 - \beta^2}$.
5. Solve $15\delta + 45 - 5\delta = 36$.
6. Solve $0.30\Omega + 0.05 = 0.25$.
7. Solve $-4x^2 + 64 = 8x - 32$.
8. Complete the square and solve: $x^2 + 14x - 14 = 0$.
9. Complete the square and solve: $1/3y^2 + 2/3y - 16 = 0$.

   Hint. Get rid of the $1/3$ first.
10. Solve using the quadratic formula: $2x^2 + 5x - 7 = 0$.
11. Solve for x.
    (a) $x^2 = 1$
    (b) $(x - 1)(x + 2) = 0$
    (c) $3x^2 - 1 = 6x + 8$

(d) $5 + 11x = -3x^2$

(e) $\sqrt{4x + 13} = x + 2$

(f) $10^{3x^2} 10^x = 100$

(g) $6x^2 - 6x - 6 = 0$

(h) $5 + 11x = -3x^2$

12. Find the inverse of $f(x) = 5x - 2$.

13. Simplify $g(f(x))$, where $f(x) = x^2 + 2$ and $g(x) = \sqrt{x - 4}$.

14. Simplify $f(g(x))$ with the same f and g. Is it the same as before?

Answer.

1. $-7\theta^2 + 21\theta - 14 = -7(\theta^2 - 3\theta + 2) = -7(\theta - 1)(\theta - 2)$.

2. $(2x - 3)(5x + 7) = 10x^2 + 14x - 15x - 21 = 10x^2 - x - 21$.

3. $q^2 - 10q + 9 = (q - 9)(q - 1)$.

4. $\dfrac{\beta - \alpha}{\alpha^2 - \beta^2} = \dfrac{\beta - \alpha}{(\alpha - \beta)(\alpha + \beta)} = \dfrac{-1}{\alpha + \beta}$.

5. $\delta = -\frac{9}{10}$.

6. $\Omega = 0.2/0.3 = 2/3$.

7. First, divide by -4: $x^2 - 16 = -2x + 8$. This is $x^2 + 2x - 24 = 0$, so the solutions are $\dfrac{-2 \pm \sqrt{4 + 4 \times 24}}{2} = \dfrac{-2 \pm 10}{2} = -1 \pm 5$, i.e., -6 and 4.

8. We have $x^2 + 14x - 14 = (x + 7)^2 - 7^2 - 14 = (x + 7)^2 - 63$, so $x^2 + 14x - 14 = 0$ is $(x + 7)^2 = 63$, or $x + 7 = \pm\sqrt{63}$, $x = -7 \pm \sqrt{63}$.

9. We have $1/3y^2 + 2/3y - 16 = 1/3[(y+1)^2 - 1] - 16 = 1/3(y+1)^2 - 49/3$, so $1/3y^2 + 2/3y - 16 = 0$ is $1/3(y + 1)^2 = 49/3$, or $y + 1 = \pm 7$, i.e., $y = -8$ or $y = 6$.

10. We have $x = \dfrac{-5 \pm \sqrt{25 + 4 \times 2 \times 7}}{2 \times 2} = \dfrac{-5 \pm 9}{4}$, so $x = -\dfrac{7}{2}$ or $x = 1$.

11. (a) We have $0 = x^2 - 1 = (x - 1)(x + 1)$, so $x = -1$ or $x = 1$.

    (b) $x = 1$ or $x = -2$.

    (c) Use the quadratic formula: $x = -1$ or $x = 3$.

    (d) Use the quadratic formula: $x = \dfrac{-11 \pm \sqrt{61}}{6}$.

(e) We square both sides and get $4x + 13 = (x+2)^2 = x^2 + 4x + 4$, so $9 = x^2$, and $x = -3$ or $x = 3$. We can verify that they work (which is not obvious since we need $4x + 13 \geqslant 0$.)

(f) This is $10^{3x^2+x} = 10^2$, so $3x^2 + x = 2$, and $x = -1$ or $x = \frac{2}{3}$.

(g) Quadratic formula: $x = \dfrac{1 \pm \sqrt{5}}{2}$.

(h) Quadratic formula: $x = \dfrac{-11 \pm \sqrt{61}}{6}$.

12. We have $5x - 2 = y$ iff $x = (y+2)/5$, so $f^{-1}(y) = (y+2)/5$.

13. We have $g(f(x)) = g(x^2 + 2) = \sqrt{x^2 + 2 - 4} = \sqrt{x^2 - 2}$.

14. We have $f(g(x)) = f(\sqrt{x-4}) = \sqrt{x-4}^2 + 2 = x - 4 + 2 = x - 2$.