Math Camp 2025 – Problem Set 1

Read the following problems carefully and justify everything you do. Avoid using calculators or computers.

1. Operations. Simplify the following expressions.

1. $\dfrac{3 \times 4}{3-2} + \dfrac{4+3}{7}$
2. $(3 \cdot 4)/(3-2) - (4+3)/7 \cdot (2+10)/3$
3. $\sum_{k=1}^{3} \left(9 + \sqrt{9^k}\right)$
4. $\prod_{x=1}^{5} (2x)$
5. $\sum_{k=1}^{n} k$
6. $\dfrac{2g+13}{3g} + \dfrac{4g-5}{4g}$
7. $\dfrac{\dfrac{w^3 z^4}{(w+1)(z-3)}}{\dfrac{(wz)^3}{(w-2)(z-3)}}$
8. $\dfrac{\prod_{i=1}^{100} 2^i}{\prod_{i=2}^{100} 2^i}$
9. $\sum_{i=1}^{N} (5^i - 5^{i-1})$.

Answer.

1. $\dfrac{3 \times 4}{3-2} + \dfrac{4+3}{7} = \dfrac{12}{1} + \dfrac{7}{7} = 12 + 1 = 13.$
2. $(3 \cdot 4)/(3-2) - (4+3)/7 \cdot (2+10)/3 = 12/1 - 7/7 \cdot 12/3 = 12 - 4 = 8.$
3. $\sum_{k=1}^{3} \left(9 + \sqrt{9^k}\right) = \sum_{k=1}^{3} \left(9 + (3^2)^{\frac{k}{2}}\right) = \sum_{k=1}^{3} \left(9 + 3^{2 \times \frac{k}{2}}\right) = \sum_{k=1}^{3} \left(9 + 3^k\right) = 9+3+9+3^2+9+3^3 = 66.$
4. $\prod_{x=1}^{5} (2x) = \prod_{x=1}^{5} 2 \times \prod_{x=1}^{5} x = 2^5 \times 5! = 32 \times 120 = 3840.$
5. A clever trick is to note that $\sum_{k=1}^{n} k = \sum_{k=1}^{n} (n-k+1)$, since it's the same sum but in the reverse order. Hence

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{1}{2}\left(\sum_{k=1}^{n} k + \sum_{k=1}^{n} k\right) = \frac{1}{2}\left(\sum_{k=1}^{n} k + \sum_{k=1}^{n} (n-k+1)\right) \\ &= \frac{1}{2}\sum_{k=1}^{n} (k + (n-k+1)) = \frac{1}{2}\sum_{k=1}^{n} (n+1) = \frac{1}{2} n(n+1).\end{aligned}$$

   You can also just guess that the answer is $\frac{1}{2}n(n+1)$ and prove it by induction. A third way is to notice that $\sum_{k=1}^{n} k$ counts the number of pairs $(a, b) \in \mathbb{N}^2$ with $1 \leqslant a < b \leqslant n+1$, which is the same as the number of 2-element subsets of $\{1, \ldots, n+1\}$, which is $\binom{n+1}{2} = \frac{1}{2}n(n+1)$.

6. $\dfrac{2g+13}{3g} + \dfrac{4g-5}{4g} = \dfrac{2g}{3g} + \dfrac{13}{3g} + \dfrac{4g}{4g} - \dfrac{5}{4g} = \dfrac{2}{3} + \dfrac{13}{3}\dfrac{1}{g} + 1 - \dfrac{5}{4}\dfrac{1}{g} = \dfrac{5}{3} + \dfrac{37}{12g}$.

7. $\dfrac{\frac{w^3z^4}{(w+1)(z-3)}}{\frac{(wz)^3}{(w-2)(z-3)}} = \dfrac{w^3z^4(w-2)(z-3)}{(wz)^3(w+1)(z-3)} = \dfrac{w^3z^4(w-2)(z-3)}{w^3z^3(w+1)(z-3)} = \dfrac{z(w-2)}{w+1}$.

8. $\dfrac{\prod_{i=1}^{100} 2^i}{\prod_{i=2}^{100} 2^i} = \dfrac{2^1 \times \prod_{i=2}^{100} 2^i}{\prod_{i=2}^{100} 2^i} = 2$.

9. $\sum_{i=1}^{N}(5^i - 5^{i-1}) = (5^1 - 5^0) + (5^2 - 5^1) + \cdots + (5^N - 5^{N-1}) = 5^N - 5^0 = 5^N - 1$.

2. Exponents and Logarithms. Simplify the following expressions assuming $x, a > 0$.

1. $x^2x^5 + x^4x^3$
2. $\dfrac{x^8}{(x^4)^2}$
3. $\dfrac{x^8}{(x^8)^4}$
4. $\sqrt[3]{1000}$
5. $\sqrt[6]{1000000}$
6. $\sqrt[3]{1000000}$
7. $\log_{10}(2x^3 5x^8)$
8. $5\log(x) - \log(x^4)$
9. $\log_4(16)$
10. $\log\left(\prod_{i=1}^{n}(ae^{x_i})\right)$

Answer.

1. $x^2x^5 + x^4x^3 = x^{2+5} + x^{4+3} = x^7 + x^7 = 2x^7$.

2. $\dfrac{x^8}{(x^4)^2} = \dfrac{x^8}{x^{4\times2}} = \dfrac{x^8}{x^{4\times2}} = \dfrac{x^8}{x^8} = 1$.

3. $\dfrac{x^8}{(x^8)^4} = \dfrac{(x^8)^1}{(x^8)^4} = (x^8)^{1-4} = (x^8)^{-3} = x^{8\times(-3)} = x^{-24}$.

4. $\sqrt[3]{1000} = \sqrt[3]{10^3} = (10^3)^{\frac{1}{3}} = 10^{3\times\frac{1}{3}} = 10^1 = 10$.

5. $\sqrt[6]{1000000} = \sqrt[6]{10^6} = (10^6)^{\frac{1}{6}} = 10^{6\times\frac{1}{6}} = 10^1 = 10$.

6. $\sqrt[3]{1000000} = \sqrt[3]{10^6} = (10^6)^{\frac{1}{3}} = 10^{6\times\frac{1}{3}} = 10^2 = 100$.

7. $\log_{10}(2x^3 5x^8) = \log_{10}(10x^{3+8}) = \log_{10}(10x^{11}) = \log_{10}(10) + \log_{10}(x^{11}) = 1 + 11\log_{10}(x)$.

8. $5\log(x) - \log(x^4) = \log(x^5) + \log(x^{-4}) = \log(x^5 x^{-4}) = \log(x^1) = \log(x)$.

9. $\log_4(16) = \log_4(4^2) = 2$.

10. $\log\left(\prod_{i=1}^{n}(ae^{x_i})\right) = \sum_{i=1}^{n}\log(ae^{x_i}) = \sum_{i=1}^{n}(\log(a) + \log(e^{x_i})) = \sum_{i=1}^{n}(\log(a) + x_i)$
$= n\log(a) + \sum_{i=1}^{n} x_i$.

3. Class Questions. Go back to the questions in Lecture 1, and make sure you can answer all of them. Write down your answers to **4.4**, **5.6**, **6** and **7.5**.

Answer.

4.4. We have

$$\begin{aligned}\prod_{i=1}^{n}\frac{x_i}{y_i} &= \frac{x_1}{y_1}\times\cdots\times\frac{x_n}{y_n}\\ &= x_1 \times \frac{1}{y_1}\times\cdots\times x_n\times\frac{1}{y_n}\\ &= x_1\times\cdots\times x_n\times\frac{1}{y_1}\times\cdots\times\frac{1}{y_n}\\ &= x_1\times\cdots\times x_n\times\frac{1}{y_1\times\cdots\times y_n}\\ &= \frac{x_1\times\cdots\times x_n}{y_1\times\cdots\times y_n}\\ &= \frac{\prod_{i=1}^{n}x_i}{\prod_{i=1}^{n}y_i}.\end{aligned}$$

The first and last equalities are the definition of $\prod_{i=1}^{n}$, and the middle ones is because $\times$ is commutative.

5.6. By definition $\log(a^x)$ is the number y such that $\exp(y) = a^x$. So we have to show that $y = x\log(a)$, i.e., that $\exp(x\log(a)) = a^x$. Now, $\exp(x\log(a)) = \exp(\log(a))^x = a^x$, as desired.

6. By definition, $\log_a(x)$ is the number y such that $a^y = x$, hence we have to show that $y = \frac{\log(x)}{\log(a)}$. Take log on both sides of $a^y = x$ and we obtain $\log(a^y) = \log(x)$, which is $y\log(a) = \log(x)$, i.e., $y = \frac{\log(x)}{\log(a)}$, as desired.

7.5. We have

$$
\begin{aligned}
&\log\left\{\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)\right]\right\} \\
&= \sum_{i=1}^{n}\log\left\{\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)\right\} \\
&= \sum_{i=1}^{n}\left\{\log\left(\frac{1}{\sqrt{2\pi}\sigma}\right)+\log\left[\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)\right]\right\} \\
&= \sum_{i=1}^{n}\left\{-\log(\sqrt{2\pi}\sigma)-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\} \\
&= -n\log(\sqrt{2\pi}\sigma)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.
\end{aligned}
$$

4. Application. The Cobb-Douglas production function relates labor (L) and capital (K) to production (Y), such that $Y = AK^\beta L^\alpha$. (The usefulness of such functions extends beyond economics; for example, Butler (2014) utilizes a Cobb-Douglas function when studying Congressional representation.) Consider that regression equations are often specified in a form such as

$$Y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \epsilon$$

where Y is the outcome, β_0 is the intercept, $\beta_1, \ldots, \beta_k$ are coefficients, $x_1, \ldots, x_k$ are the independent variables, and ϵ is an error term. Without worrying about the error term, manipulate the Cobb-Douglas production function so that it is in such a form, where β and α are the coefficients.

Hint. A variable in a regression may actually be a "transformed" variable; for example, for various reasons a researcher with one independent variable x_1 may choose to estimate an effect β_1 using $Y = \beta_0 + \beta_1\sqrt{x_1}$ rather than $Y = \beta_0 + \beta_1 x_1$, though you should note the coefficient's interpretation is changed.

Answer. Take logs to $Y = AK^\beta L^\alpha$:

$$\log(Y) = \log(A) + \beta\log(K) + \alpha\log(L).$$

This is a linear equation with outcome $\log(Y)$, independent variables $\log(K)$ and $\log(L)$, intercept $\log(A)$, and coefficients β and α.