Lecture 7: Square Matrices

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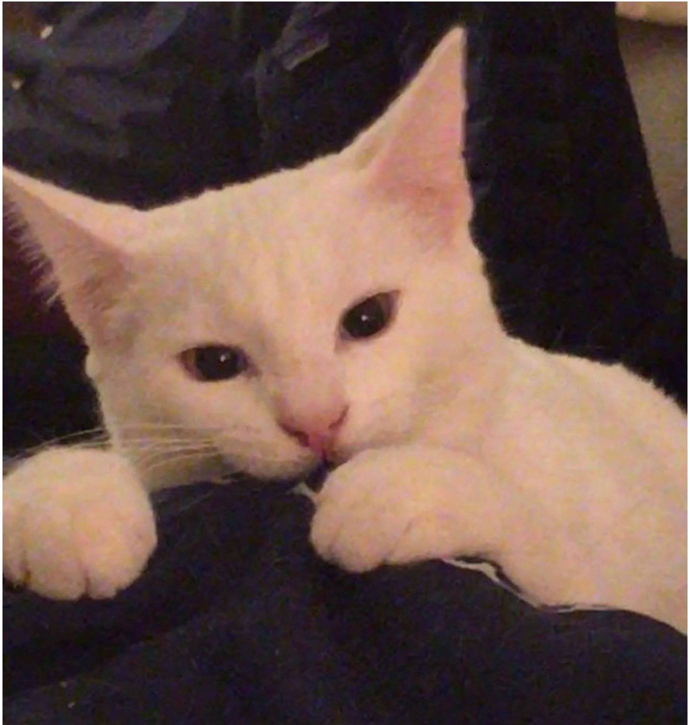
Plan

Square matrices

1. Identity, diagonal and symmetric matrices
2. Gaussian elimination
3. Inverse
4. Trace and determinant



Square Matrices

Consider $\mathbb{R}^{n\times n}$, the set of n by n **square** matrices.

If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ then $\boldsymbol{A}+\boldsymbol{B}$, $c\boldsymbol{A}$ with $c \in \mathbb{R}$ and $\boldsymbol{AB}$ are also in $\mathbb{R}^{n\times n}$. Also $\boldsymbol{A}^\top \in \mathbb{R}^{n\times n}$.

We define the **identity matrix** $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ as the matrix

$$\boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We have $\boldsymbol{I}_n\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n = \boldsymbol{A}$ for every $\boldsymbol{A} \in \mathbb{R}^{n\times n}$.

Diagonal and Triangular Matrices

Given $a_1, \dots, a_n \in \mathbb{R}$ we define the **diagonal matrix**

$$\mathbf{diag}(\boldsymbol{a_1, \dots, a_n}) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

We say that $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is **lower/upper triangular** if it has zeros above/below the diagonal:

$$\underset{\text{lower triangular}}{\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \qquad \underset{\text{upper triangular}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}}$$

Symmetric Matrices

We say that $\boldsymbol{A}$ is **symmetric** if $\boldsymbol{A}^\top = \boldsymbol{A}$.

In other words, if the rows of $\boldsymbol{A}$ equal the columns of $\boldsymbol{A}$.

Examples.

- $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric.
- $\begin{pmatrix} 1 & 2 & \mathbf{3} \\ 2 & 4 & 5 \\ \mathbf{7} & 5 & 6 \end{pmatrix}$ is not.

Question 1

Convince yourself that if $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ then $\boldsymbol{A}^\top \boldsymbol{A}$ is symmetric.

Gaussian Elimination

We want to solve a system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$.

Idea: transform $\boldsymbol{A}$ into an upper triangular matrix using transformations that are equivalent to left-multiplying by matrices $\boldsymbol{C}$. If we do the same to $\boldsymbol{b}$, we preserve the equation: $\boldsymbol{Ax} = \boldsymbol{b}$ implies $\boldsymbol{CAx} = \boldsymbol{Cb}$.

We can:

- Take one row and multiply it by a nonzero scalar.
- Take one row and add it to another one, possibly multiplied by some scalar.
- Exchange two rows.

Once A is upper triangular, the system of equations is easy.

Gaussian Elimination

We want to solve

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let's clean the first column. Take the first row and subtract it from the second one. Let's keep track of the changes, ignoring $\boldsymbol{x}$ for now:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right).$$

Let's clean the second column. We want to use the second row, so let's first divide it by -2:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

Now subtract it from the third row.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 1 \end{array}\right).$$

Great! Now the equation is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{i.e.,} \qquad \begin{cases} x_1 + x_2 = 1 \\ x_2 - \dfrac{1}{2}x_3 = 0 \\ \dfrac{3}{2}x_3 = 1 \end{cases}$$

so $x_3 = \frac{2}{3}$, $x_2 = \frac{1}{2}x_3 = \frac{1}{3}$, and $x_1 = 1 - x_2 = \frac{1}{3}$.

Example. We want to solve $\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 - x_3 = 0 \\ \quad\quad x_2 + x_3 = 1 \end{cases}$.

Gauss Algorithm:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{array}\right)$$

Notice. We had to exchange rows in the second step.

The system becomes

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \quad\quad x_2 + x_3 = 1 \\ \quad\quad\quad -2x_3 = -1 \end{cases},$$

so $x_3 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, and $x_1 = 1 - x_2 - x_3 = 0$.

Exercise

Question 2

Solve

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 - x_3 = 0 \\ 2x_1 \qquad\quad + x_3 = 1. \end{cases}$$

using Gaussian elimination.

Inverse

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ we say that $\boldsymbol{A}^{-1} \in \mathbb{R}^{n \times n}$ is the **inverse** of $\boldsymbol{A}$ if

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n.$$

Not every nonzero matrix has an inverse. For example, suppose that

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has an inverse $\boldsymbol{A}^{-1}$. We have $\boldsymbol{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$.

Multiplying by $\boldsymbol{A}^{-1}$ the LHS we get: $\boldsymbol{A}^{-1}\boldsymbol{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boldsymbol{I}_n\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. But multiplying the RHS we get $\boldsymbol{A}^{-1}\mathbf{0} = \mathbf{0} \neq \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This is impossible. Therefore, $\boldsymbol{A}^{-1}$ cannot exist.

Properties of the Inverse

If $\boldsymbol{A}$ has an inverse we say that $\boldsymbol{A}$ is **invertible**.

Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are invertible. Then

- The inverse is unique.
- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$.
- $(c\boldsymbol{A})^{-1} = c^{-1}\boldsymbol{A}^{-1}$ if $c \in \mathbb{R}$, $c \neq 0$.
- $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
- $(\boldsymbol{A}^{-1})^\top = (\boldsymbol{A}^\top)^{-1}$.
- $\operatorname{diag}(a_1, \ldots, a_n)^{-1} = \operatorname{diag}(a_1^{-1}, \ldots, a_n^{-1})$ if $a_1, \ldots, a_n \in \mathbb{R} \setminus \{0\}$.

Question 3

Why?

Calculating the Inverse

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ we want $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$.

We can use Gauss Algorithm: if we do transformations to $\boldsymbol{A}$ that are left-multiplying by $\boldsymbol{C}$, and we end up with $\boldsymbol{C}\boldsymbol{A} = \boldsymbol{I}_n$, then $\boldsymbol{A}^{-1} = \boldsymbol{C} = \boldsymbol{C}\boldsymbol{I}_n$, so $\boldsymbol{A}^{-1}$ is the same as doing those transformations to $\boldsymbol{I}_n$.

Example. We want $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$$

So $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

Example. Let's try to find $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right) \Rightarrow ?$$

If we get an upper triangular matrix with a zero in the diagonal, we won't be able to invert it.

Exercise

Question 4

Calculate $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$ if it exists.

Question 5

Calculate $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$ if it exists.

Exercise

Question 6

Verify that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $ad - bc \neq 0$.

What happens if $ad - bc = 0$?

Trace

The **trace** of a square matrix is the sum of its diagonal elements. It defines the function $\mathrm{tr} : \mathbb{R}^{n \times n} \to \mathbb{R}$ given be

$$\mathrm{tr}(\boldsymbol{A}) = a_{11} + \cdots + a_{nn}.$$

Example. $\mathrm{tr} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -1 \end{pmatrix} = 1 + 2 - 1 = 2.$

Properties.

- $\mathrm{tr}(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B})$,
- $\mathrm{tr}(c\boldsymbol{A}) = c\,\mathrm{tr}(\boldsymbol{A})$ if $c \in \mathbb{R}$,
- $\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{B}\boldsymbol{A})$ even if $\boldsymbol{A}, \boldsymbol{B}$ are not square, as long as $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$ are,
- $\mathrm{tr}(\boldsymbol{A}^\top) = \mathrm{tr}(\boldsymbol{A})$.

Exercise

Question 7
Convince yourself that $\mathrm{tr}(\boldsymbol{A}+\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{B})$ and $\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}) = \mathrm{tr}(\boldsymbol{B}\boldsymbol{A})$.

Determinant

The **determinant** of a square matrix A, $\mathbf{det}(\boldsymbol{A})$, is defined recursively as follows:

If $\boldsymbol{a} \in \mathbb{R}^{1\times 1}$ then $\det(\boldsymbol{a}) = \boldsymbol{a}$.

Given

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

we take the first column, and for each cell a_{i1} we delete the row and column of that cell, obtaining a matrix $\boldsymbol{M}_{i1}$. The determinant is then

$$\det(\boldsymbol{A}) = a_{11}\det(\boldsymbol{M}_{11}) - a_{21}\det(\boldsymbol{M}_{21}) + \cdots + (-1)^{n+1}a_{n1}\det(\boldsymbol{M}_{n1}).$$

We can write $|\boldsymbol{A}|$ instead of $\det(\boldsymbol{A})$.

Example

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -1 \end{pmatrix} = 1\cdot\det\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + (-1)\cdot\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$$

Now,

- $\det\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} = 2\cdot(-1) - 1\cdot(-1) = -2 + 1 = -1,$
- $\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 1\cdot(-1) - 1\cdot 1 = -2,$ and
- $\det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = 1\cdot(-1) - 2\cdot 1 = -1 - 2 = -3.$

Therefore, $\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -1 \end{pmatrix} = 1\cdot(-1) - 1\cdot(-2) + (-1)\cdot(-3) = -1 + 2 + 3 = 4.$

Diagonal and Triangular Matrices

Examples.

- $\det\begin{pmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} = 2\cdot\det\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 2\cdot 1\cdot\det(-1) = 2\cdot 1\cdot(-1) = -2.$
- $\det\begin{pmatrix}2 & 5 & 3\\ 0 & 1 & 7\\ 0 & 0 & -1\end{pmatrix} = 2\cdot\det\begin{pmatrix}1 & 7\\ 0 & -1\end{pmatrix} = 2\cdot 1\cdot\det(-1) = 2\cdot 1\cdot(-1) = -2.$

In general, if $\boldsymbol{A}$ is diagonal or triangular then $\det(\boldsymbol{A})$ is the product of the numbers in the diagonal.

Exercise

Question 8

Calculate $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

Properties

If $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n \times n}$ and $c \in \mathbb{R}$ then

- $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$,
- $\det(\boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{A}) \det(\boldsymbol{B})$,
- $\det(\boldsymbol{I}_n) = 1$,
- $\boldsymbol{A}$ is invertible iff $\det(\boldsymbol{A}) \neq 0$; in that case, $\det(\boldsymbol{A}^{-1}) = \det(\boldsymbol{A})^{-1}$,
- $\det(\boldsymbol{A}^\top) = \det(\boldsymbol{A})$.

Adjoint Matrix

Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ let M_{ij} be the determinant of the matrix that results from removing the ith row and the jth column of $\boldsymbol{A}$.

The **adjoint** of $\boldsymbol{A}$ is the n by n matrix **adj($\boldsymbol{A}$)** whose ij entry is $(-1)^{i+j}M_{ji}$.

Example.

$$\operatorname{adj}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} \det\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \\ -\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ \det\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \det\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \end{pmatrix}^{\top} = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 0 & -2 \\ -3 & 2 & 1 \end{pmatrix}^{\top}$$

Key Property. $\operatorname{adj}(\boldsymbol{A})\boldsymbol{A} = \det(\boldsymbol{A})\boldsymbol{I}_n$, so $\boldsymbol{A}^{-1} = \det(\boldsymbol{A})^{-1}\operatorname{adj}(\boldsymbol{A})$.

Exercise

Question 9

Use the formula $\boldsymbol{A}^{-1} = \det(\boldsymbol{A})^{-1}\text{adj}(\boldsymbol{A})$ to calculate $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}$.