Lecture 5: Integrals

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Plan

Quick review of last lecture

Integrals

1. Definition
2. Fundamental Theorem of Calculus
3. Indefinite Integrals
4. Improper Integrals

Review of Last Lecture

How to calculate derivatives. If a is a constant, and u, v are functions of x, we have:

- $a' = 0$
- $(x^k)' = kx^{k-1}$
- $(e^x)' = e^x$
- $\log'(x) = \frac{1}{x}$
- $(au)' = au'$
- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + uv'$
- $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

Chain rule: $(f(g(x)))' = f'(g(x))g'(x)$.

Example. We know $\log'(x) = \frac{1}{x}$ and $(x^2+1)' = 2x$. Therefore,

$$(\log(x^2+1))' = \log'(x^2+1) \cdot (x^2+1)' = \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}.$$

Review of Last Lecture

Example. We know $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$ and $(x^2+1)' = 2x$. Therefore,

$$(\sqrt{x^2+1})' = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

Example. We know $(e^x)' = e^x$ and $(x^2+1)' = a$. Therefore, $(e^{x^2+1})' = e^{x^2+1} \cdot 2x$.

It's good to know that you can calculate any derivative. In practice your computer will do this for you. E.g., ML and Bayesian stats libraries (PyTorch, Stan) do automatic differentiation, i.e., they calculate derivatives for you. But you need to understand what the computer is doing.

Convexity and Optimization

- Increasing if $f' > 0$. Decreasing if $f' < 0$.

  If f increases up to x_0 and then decreases, it has a maximum a x_0.

- Concave if $f'' \leqslant 0$. Convex if $f'' \geqslant 0$.

  If f is concave and $f'(x_0) = 0$, f is maximized at $x = x_0$.

  If $f''(x_0) < 0$, f is locally concave, so if $f'(x_0) = 0$ then f has a local maximum.

  Some functions are convex/concave everywhere. Others are convex on some parts of the domain and concave on others.

  E.g., $f(x) = x^3$, $f''(x) = 6x$, so f is concave on $(-\infty, 0]$ and convex on $[0, +\infty)$.

- Not every function has local extrema (minima/maxima).

  E.g., $f : [0,1] \to \mathbb{R}$, $f(x) = x$ is increasing, so it's minimized at $x = 0$ and maximized at $x = 1$, and it doesn't have local minima/maxima.

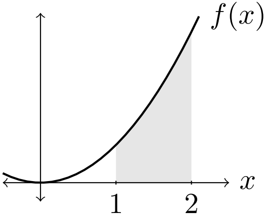
Integrals

If $f : [a, b] \to [0, +\infty]$ we define the **integral** of f as the area below the graph of f, and denote it by

$$\int_a^b f(x)\,dx.$$

Take $f(x) = x^2$, $a = 1$, $b = 2$.

$\int_1^2 x^2\,dx$ is the gray area.

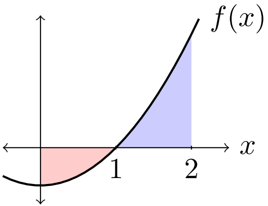


Integrals

If $f : [a, b] \to \mathbb{R}$ we can define the integral $\int_a^b f(x)\,dx$ by subtracting the area below the graph when $f \geqslant 0$ with the area above the graph when $f \leqslant 0$.

Take $f(x) = x^2 - 1$, $a = 0$, $b = 2$.

$\int_0^2 (x^2 - 1)\,dx$ is the blue minus the red area.



If $a < b$ we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Fundamental Theorem of Calculus

Theorem

Given $f : [a, b] \to \mathbb{R}$ we can define $F : [a, b] \to \mathbb{R}$ given by

$$F(x) = \int_a^x f(t)\,dt.$$

If f is continuous we have $F'(x) = f(x)$ for all $x \in [a, b]$.

Why? We have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\left\{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right\} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since f is continuous, $f(t)$ in $[x, x+h]$ is very close to $f(x)$, hence $\int_x^{x+h} f(t)\,dt$ should be close to $\int_x^{x+h} f(x)\,dt = hf(x)$. Therefore $\frac{1}{h}\int_x^{x+h} f(t)\,dt \approx \frac{1}{h}hf(x) = f(x)$, so when $h \to 0$ we should get $\frac{1}{h}\int_x^{x+h} f(t)\,dt \to f(x)$.

Calculating Integrals

Take $f(x) = x^2$ and $F(x) = \int_1^x f(t)\,dt$. We know that $F'(x) = f(x) = x^2$.

What can F be? The function $g(x) = \frac{1}{3}x^3$ satisfies $g'(x) = f(x)$. And $(F-g)'(x) = F'(x) - g'(x) = 0$, so $F - g$ must be a constant. (Why?)

Therefore $F(x) = \frac{1}{3}x^3 + c$ for some $c \in \mathbb{R}$.

We know $F(1) = \int_1^1 f(t)\,dt = 0$, so $F(1) = \frac{1}{3}1^3 + c = 0$, i.e., $c = -\frac{1}{3}$ and $F(x) = \frac{1}{3}x^3 - \frac{1}{3}$.

In particular, $\displaystyle\int_1^2 f(t)\,dt = F(2) = \frac{1}{3}(2^3 - 1) = \frac{7}{3}$.

In general, if $F' = f$ we must have $\boxed{\displaystyle\int_a^b f(x)\,dx = F(b) - F(a).}$

Exercise

Question 2

Calculate $\int_0^1 (x^2 + e^x)\,dx$.

Indefinite Integrals

To calculate the integral of f we need to find a function F whose derivative is f. We call F an **antiderivative** or **indefinite integral** of f and denote it by

$$\int \boldsymbol{f(x)\,dx}.$$

By definition $\int F'(x)\,dx = F(x)$.

Since $(F+G)' = F' + G' = f + g$ and $(aF)' = aF' = af$ if $a \in \mathbb{R}$, we have

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx \quad \text{and}$$
$$\int af(x)\,dx = a\int f(x)\,dx \quad \text{if } a \in \mathbb{R}.$$

Basic Integrals

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c$ for any $c \in \mathbb{R}$ if $n \neq -1$.

So, for example, $\int (x^2 - x + 1) \, dx = \dfrac{x^3}{3} - \dfrac{x^2}{x} + x + c$ for any c.

If $n = -1$, $\int \dfrac{1}{x} \, dx = \log(x) + c$.

$\int e^{ax} \, dx = \dfrac{1}{a} e^{ax} + c$.

Integration by Parts

Integrating $(fg)' = f'g + fg'$ we obtain

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

In particular,

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

Example.

$$\begin{aligned}\int \log(x)\,dx &= \int x' \log(x)\,dx \\ &= x\log(x) - \int x \log'(x)\,dx \\ &= x\log(x) - \int x\frac{1}{x}\,dx \\ &= x\log(x) - \int 1\,dx = x\log(x) - x + c.\end{aligned}$$

Exercise

Question 3

Find $\displaystyle\int xe^x \, dx$.

More Integrals

Integrating the chain rule $f(g(x))' = f'(g(x))g'(x)$ we obtain

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + c.$$

Examples: $\int e^{f(x)} f'(x)\,dx = e^{f(x)} + c$ and $\int \frac{f'(x)}{f(x)}\,dx = \log(f(x)) + c$.

So, for example, $\int \frac{2x}{x^2+1}\,dx = \log(x^2+1)$ and $\int e^{-x^2} x\,dx = -\frac{1}{2}e^{-x^2}$.

In general there isn't an "elementary" formula for the integral of any elementary function. For example, there is no elementary formula for $\int e^{-x^2}\,dx$. (Search "Nonelementary integral" in Wikipedia.)

Exercise

Question 4

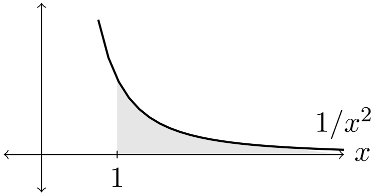
Calculate the following integrals:

1. $\int_{-1}^{1} (2x + 1)\, dx,$
2. $\int_{0}^{1} \sqrt{x}\, dx,$
3. $\int_{1}^{e} \frac{1}{x}\, dx,$
4. $\int_{0}^{1} x \log(x)\, dx.$

Improper Integrals

We have $\displaystyle\int_1^x \frac{1}{t^2}\,dt = \left.\frac{-1}{t}\right|_1^x = \frac{-1}{x} - \frac{-1}{1} = 1 - \frac{1}{x}.$

Hence $\displaystyle\lim_{x\to+\infty}\int_1^x \frac{1}{t^2}\,dt = \lim_{x\to+\infty}\left(1-\frac{1}{x}\right) = 1.$



In general we define the **improper integrals**

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx,$$
$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx,$$
$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to+\infty}\int_{-a}^a f(x)\,dx.$$

Exercise

Question 5

Calculate the following integrals for any $\lambda > 0$:

1. $\displaystyle\int_0^{+\infty} e^{-\lambda x}\,dx$,
2. $\displaystyle\int_{-\infty}^{+\infty} e^{-\lambda |x|}\,dx$.