Lecture 3: Limits

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Plan

Limits

1. Definition
2. Calculating limits
3. Lateral limits
4. Continuity

Sequences

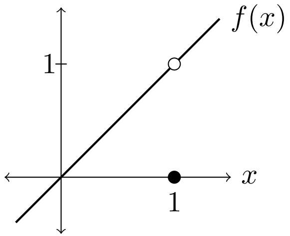
1. Convergence
2. Subsequences, monotonicity

Good Morning!

Limits

Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}.$$



We have $f(1) = 0$. But if we look at $f(x)$ for x close to 1, excluding $x = 1$, we see that $f(x)$ gets close to 1, not 0.

We denote this by saying that the **limit** of f when x tends to 1 is 1, or

$$\lim_{x \to 1} f(x) = 1.$$

Semi-formal Definition

If $f : X \to \mathbb{R}$ is a function, where $X \subset \mathbb{R}$ (i.e., X is a set of real numbers), and $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, we say that

$$\lim_{\boldsymbol{x \to x_0}} \boldsymbol{f(x)} = L$$

if there are numbers $x \in X$ arbitrarily close to x_0 but not equal to x_0 and, as those x get arbitrarily close to x_0, $f(x)$ gets arbitrarily close to the limit L.

Properties of Limits

If $\lim\limits_{x\to x_0} f(x) \in \mathbb{R}$ and $\lim\limits_{x\to x_0} g(x) \in \mathbb{R}$ we have

$$\lim_{x\to x_0}(f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x),$$

$$\lim_{x\to x_0}(f(x)g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x),$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)} \quad \text{if } \lim_{x\to x_0} g(x) \neq 0.$$

Example.

$$\lim_{x\to 1} \frac{x+1}{x} = \frac{\lim\limits_{x\to 1}(x+1)}{\lim\limits_{x\to 1} x} = \frac{\lim\limits_{x\to 1} x + \lim\limits_{x\to 1} 1}{\lim\limits_{x\to 1} x} = \frac{1+1}{1} = 2.$$

Infinite Limits

A limit can be infinite, e.g., $\lim_{x \to +\infty} x = +\infty$.

We say that $\boldsymbol{f(x) \to L}$ as $x \to x_0$ if $\lim_{x \to x_0} f(x) = L$.

We say that $\boldsymbol{f(x) \to \infty}$ if $|f(x)| \to +\infty$.

Example. $\dfrac{1}{x} \to \infty$ as $x \to 0$.

In general, if $f(x) \to 0$ then $\dfrac{1}{f(x)} \to \infty$.

If $f(x) \to \infty$ then $\dfrac{1}{f(x)} \to 0$.

Arithmetic of Infinite Limits

1. If $f(x) \to +\infty$ and $g(x) \to L \in \mathbb{R} \cup \{+\infty\}$ then $f(x) + g(x) \to +\infty$.

   Example. If $x \to 0$ then $\frac{1}{x^2} \to +\infty$ and $e^x \to 1$, so $\frac{1}{x^2} + e^x \to +\infty$.

2. If $f(x) \to \infty$ and $g(x) \to L \in [-\infty, +\infty] \setminus \{0\}$ then $f(x)g(x) \to \infty$.

   Example. If $x \to 0$ then $\frac{1}{x} \to \infty$ and $e^x \to 1$, so $\frac{e^x}{x} \to \infty$.

   We can determine the sign in the natural way.

   Example. If $x \to 0$ then $\frac{1}{x^2} \to +\infty$ and $x - 1 \to -1$, so $(x-1)\frac{1}{x^2} \to -\infty$.

“Indeterminate” Limits

$(+\infty) - (+\infty)$ [or in general $\infty \pm \infty$], $0 \times \infty$, $\dfrac{\infty}{\infty}$ and $\dfrac{0}{0}$ can be anything, and may not even exist.

Examples.

1. If $x \to +\infty$, $(x+1) - x = 1 \to 1$. But $(2x) - x = x \to +\infty$.
2. If $x \to 0$, $x \cdot \dfrac{1}{x} = 1 \to 1$, but $x^2 \cdot \dfrac{1}{x} = x \to 0$.
3. If $x \to \infty$, $\dfrac{x}{x} = 1 \to 1$, but $\dfrac{x^2}{x} = x \to \infty$.
4. If $x \to 0$, $\dfrac{x}{x} = 1 \to 1$, but $\dfrac{x^2}{x} = x \to 0$ and $\dfrac{x}{x^2} = \dfrac{1}{x} \to \infty$.

Two Examples

Take $x \to +\infty$. We have $\underbrace{x^2}_{\to +\infty} - \underbrace{x}_{\to +\infty}$ so it's not obvious what happens.

But

$$x^2 - x = \underbrace{x^2}_{\to +\infty} \underbrace{\left(1 - \frac{1}{x}\right)}_{\to 1} \to +\infty.$$

Consider

$$\frac{\overbrace{x-1}^{\to +\infty}}{\underbrace{2x+1}_{\to +\infty}} \to ?$$

We have

$$\frac{x-1}{2x+1} = \frac{x\left(1 - \frac{1}{x}\right)}{x(2 + \frac{1}{x})} = \frac{1 - \frac{1}{x}}{2 + \frac{1}{x}} \to \frac{1}{2}.$$

Question 1

Find the following limits.

1. $\lim_{x\to-\infty} x^2$
2. $\lim_{x\to-\infty} (x^3 + x^2)$
3. $\lim_{x\to-\infty} \exp(x)$
4. $\lim_{x\to 0} \log(x)$
5. $\lim_{x\to+\infty} \frac{2x^3 - x + 1}{x^2 + x + 2}$
6. $\lim_{x\to 2} \frac{x^3 - 8}{x - 2}$

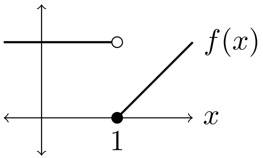
Hint. For the last one, use that for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

Lateral Limits

Consider this function:

$$f : \mathbb{R} \to \mathbb{R},$$
$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ x - 1 & \text{if } x \geqslant 1. \end{cases}$$



If $x \to 1$ with $x < 1$ then $f(x) = 1$, so $f(x) \to 1$. In that case we say that the **limit from the left** of f at 1 is 1, or $\lim\limits_{x\to 1^-} f(x) = 1$.

If $x \to 1$ with $x > 1$ then $f(x) = x - 1$, so $f(x) \to 0$. In that case we say that the **limit from the right** of f at 1 is 1, or $\lim\limits_{x\to 1^+} f(x) = 0$.

If $\lim\limits_{x\to x_0} f(x)$ exists then the lateral limits have to be equal to it.

Since in this case the left and right limits don't coincide, the limit $\lim\limits_{x\to 1} f(x)$ doesn't exist.

Question 2

Find the following limits.

1. $\lim_{x \to 0^-} \frac{1}{x}$
2. $\lim_{x \to 0^+} \frac{1}{x}$
3. $\lim_{x \to 0^-} \frac{1}{x^2}$
4. $\lim_{x \to 0^+} \frac{1}{x^2}$
5. $\lim_{x \to 0^+} \log(x)$.

Continuity

If $f : X \to \mathbb{R}$ with $X \subset \mathbb{R}$ and $x_0 \in X$ we say that f is **continuous at x_0** if for $x \in X$ close to x_0 we have that $f(x)$ is arbitrarily close to $f(x_0)$.

We say that $f : X \to \mathbb{R}$ is **continuous** if it's continuous at each $x_0 \in X$.

(We'll define this properly later.)

All the functions we've seen so far are continuous. If f, g are continuous at x_0 then $f + g$, fg, cf for $c \in \mathbb{R}$, and f/g if $g(x_0) \neq 0$ are also continuous at x_0.

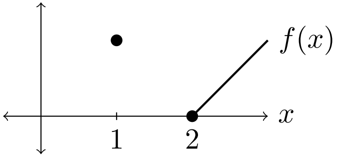
If f is continuous at x_0 and g is continuous at $f(x_0)$ then $g(f(x))$ is continuous at x_0.

If there are points around x_0 in the domain of f then continuity at x_0 means that $\lim_{x \to x_0} f(x) = f(x_0)$.

Visual Examples

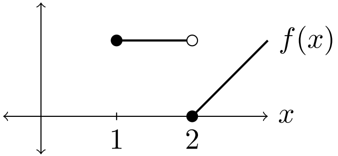
This is continuous:

$$f : \{1\} \cup [2, +\infty) \to \mathbb{R},$$
$$f(x) = \begin{cases} 1 & \text{if } x = 1, \\ x - 2 & \text{if } x \geqslant 2. \end{cases}$$



This is not continuous:

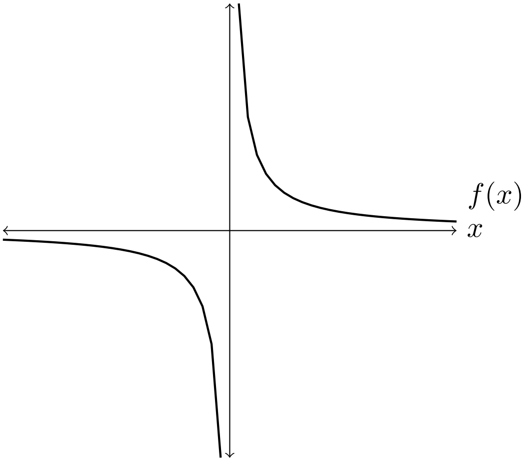
$$f : [1, +\infty) \to \mathbb{R},$$
$$f(x) = \begin{cases} 1 & \text{if } x \in [1, 2), \\ x - 2 & \text{if } x \geqslant 2. \end{cases}$$



We have $\lim\limits_{x \to 2^-} f(x) = 1$ but $f(2) = 0$. $\lim\limits_{x \to 2} f(x)$ doesn't exist.

Question 3

Is $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = 1/x$ continuous?



Sequences

A **sequence** of real numbers is a an infinite list $x_1, x_2, \ldots \in \mathbb{R}$. We can write it as $\{x_n\}_{n \in \mathbb{N}}$. We can think of it as a function $x : \mathbb{N} \to \mathbb{R}$.

We say that the sequence $\{x_n\}_{n \in \mathbb{N}}$ **converges** to $L \in \mathbb{R}$, or that its **limit** is L if, as n grows, x_n gets arbitrarily close to L. We write $x_n \to L$ or

$$\lim_{n \to \infty} x_n = L.$$

Example. $\displaystyle \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{n(2+\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}.$

> **Question 4**
>
> Find $\displaystyle \lim_{n\to\infty} \frac{2^n}{n!}$. Here $n! = 1 \times \cdots \times n$, e.g., $3! = 1 \times 2 \times 3 = 6$.

A Better Definition of Limits

Recall what I wrote above:

> If $f : X \to \mathbb{R}$ is a function, where $X \subset \mathbb{R}$, and $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, we say that
>
> $$\lim_{x \to x_0} f(x) = L$$
>
> if there are numbers $x \in X$ arbitrarily close to x_0 but not equal to x_0 and, as those x get arbitrarily close to x_0, $f(x)$ gets arbitrarily close to the limit L.

A formal definition that makes that precise:

If $f : X \to \mathbb{R}$ is a function, where $X \subset \mathbb{R}$, and $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, we say that

$$\lim_{x \to x_0} f(x) = L$$

if (a) there is a sequence $x_n \to x_0$ with $x_n \in X \setminus \{x_0\}$, and
(b) for every sequence $x_n \to x_0$ with $x_n \in X \setminus \{x_0\}$ we have that $f(x_n) \to L$.

Subsequences

A **subsequence** of $\{x_n\}$ is any sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where $\{n_k\}$ is an increasing sequence of natural numbers.

Theorem

If $\lim_{n\to\infty} x_n = L$ and $\{x_{n_k}\}$ is a subsequence then $\lim_{k\to\infty} x_{n_k} = L$.

For example, if $x_n = (-1)^n$ then

$x_{2k} = (-1)^{2k} = 1$ is a subsequence, and

$x_{2k+1} = (-1)^{2k+1} = -1$ is another one.

$$-1,\ 1,\ -1,\ 1,\ -1,\ 1,\ \dots$$

Since $x_{2k} \to 1$ but $x_{2k+1} \to -1$, x_n is not convergent.

Monotone Sequences

A sequence $\{x_n\}$ is **monotone** if either $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$ or $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$.

For example, $x_n = (-1)^n$ is not monotone.

We say that $\{x_n\}$ is **monotone non-decreasing** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$, and **monotone non-increasing** if $x_n \geqslant x_{n+1}$ for all $n \in \mathbb{N}$.

We say that $\{x_n\}$ is **bounded** if there is a number $C \in \mathbb{R}$ such that $|x_n| \leqslant C$ for all $n \in \mathbb{N}$. E.g., $x_n = n$ is not bounded, but $x_n = 1/n$ is bounded.

Theorem

If $\{x_n\}$ is monotone and bounded then it converges.

Example

Take $x_1 = 1$ and $x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$.

We have

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \geqslant 2\sqrt{\frac{x_n}{2}\frac{1}{x_n}} = 2\sqrt{\frac{1}{2}} = \sqrt{2},$$

so $x_n^2 \geqslant 2$ for any $n \geqslant 2$. (I have used that for all $x, y \geqslant 0$, $x + y \geqslant 2\sqrt{xy}$.)

If $n \geqslant 2$ we have

$$x_{n+1} - x_n = \frac{1}{x_n} - \frac{x_n}{2} = \frac{2 - x_n^2}{2x_n} \leqslant 0.$$

Therefore $\{x_n\}$ is monotone non-increasing starting at $n = 2$. Clearly $x_n \geqslant 0$, and $x_n \leqslant x_2 = 1.5$, so $|x_n| \leqslant 1.5$, and x_n is bounded. By the Theorem, x_n converges to some $L \in \mathbb{R}$.

We have $x_{n+1} \to L$ and $\dfrac{x_n}{2} + \dfrac{1}{x_n} \to \dfrac{L}{2} + \dfrac{1}{L}$, hence $L = \dfrac{L}{2} + \dfrac{1}{L}$, i.e., $L^2 = 2$. Since $x_n \geqslant 0$, $L \geqslant 0$, so $L = \sqrt{2}$.