Lecture 2: Sets and Functions

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Plan

Sets

1. Extension and Comprehension
2. Operations
3. Intervals

Functions

1. Domain and Range
2. Inverse functions

Good Morning!

Sets

A **set** is a collection of things.

If A is a set and it contains x, we say that $\boldsymbol{x \in A}$. We say that x is an **element** of A.

Two sets A and B are the same (i.e., $A = B$) if and only if they have the same elements. In other words, if for every x, $x \in A$ if and only if $x \in B$.

We can define a set by extension, i.e., by listing its elements, e.g.,

$$A = \{2, \pi, \text{Juan}\}$$

is the set that contains the numbers 2 and π, and it also contains me.

The **empty set** contains nothing: $\varnothing = \{\}$.

Sets can contain other sets, e.g., $\{\varnothing, \{\varnothing\}\}$ is a set. In fact, in modern math (with the Zermelo-Fraenkel axioms) everything is a set, so there is nothing else to contain.

Set Comprehension

We can define a set by comprehension: is A is a set, we can define the set

$$\{\boldsymbol{x \in A : P(x)}\} \quad \text{or, equivalently,} \quad \{\boldsymbol{x : x \in A, P(x)}\}$$

where $P(x)$ is a property. For example,

$$\{n \in \mathbb{N} : n < 10 \text{ and } n \text{ is odd}\} = \{1, 3, 5, 7, 9\}.$$

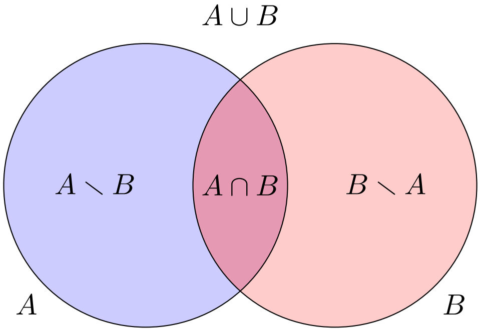
If $A = \{2, \pi, \text{Juan}\}$,

$$\{x \in A : x \text{ is a person}\} = \{\text{Juan}\}.$$

Sometimes people write $\{x \in A \mid P(x)\}$.

Operations on Sets

If A and B are sets we can define their **union**, $\boldsymbol{A \cup B}$, which is the set that contains the elements of A and those of B.



We can define their **intersection** as

$$\boldsymbol{A \cap B} = \{x \in A : x \in B\},$$

and their **difference** as

$$\boldsymbol{A \setminus B} = \{x \in A : x \notin B\}.$$

We have

- $x \in A \cup B$ if and only if $x \in A$ **or** $x \in B$.
- $x \in A \cap B$ if and only if $x \in A$ **and** $x \in B$.
- $x \in A \setminus B$ if and only if $x \in A$ **and** $x \notin B$.

Number Intervals

If $a, b \in \mathbb{R}$ we define the **closed interval**

$$[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\},$$

the **open interval**

$$(a, b) = \{x \in \mathbb{R} : a < x < b\},$$

and the interval

$$[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}.$$

Also $[a, +\infty) = \{x \in \mathbb{R} : a \leqslant x\}$, $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$ and so on.

Question 1

Write the following sets as intervals:

1. $[0, 2] \cap [1, 3]$,
2. $[0, 2] \cup [1, 3)$,
3. $[0, 2] \setminus [1, 3]$.

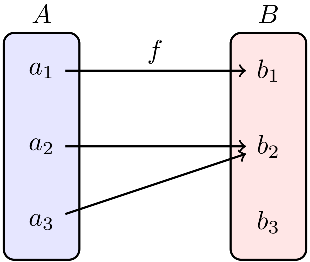
Functions

If A and B are two sets a **function $f : A \to B$** is a mapping of each element of A to an element of B. If $x \in A$, f maps it to $\boldsymbol{f(x)} \in B$.

We call A the **domain** of f, and B its **codomain**. The **range** of f is

$$\boldsymbol{f(A)} = \{f(x) : x \in A\}.$$

Example:



Here we have $f(a_1) = b_1$, and $f(a_2) = f(a_3) = b_2$. The range is $f(A) = \{b_1, b_2\}$.

Question 2

Suppose that $f : D \to \mathbb{R}$ is a function that satisfies the following equation for all $x \in D$. Find the maximal domain D.

1. $f(x) = \dfrac{1}{x}$,
2. $f(x) = \dfrac{2x - 1}{\log(x - 1)}$,
3. $f(x) = \sqrt{\exp(x) - 1}$,
4. $f(x) = \log(x) + \log(1 - x)$.

Some Functions

We saw the functions $\exp : \mathbb{R} \to (0, +\infty)$ and $\log : (0, +\infty) \to \mathbb{R}$.

Linear functions are $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ for some $a, b \in \mathbb{R}$.

Quadratic functions are $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$ with $a \neq 0$.

Polynomial functions are $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for some $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$.

Example: $f(x) = 2x^5 - 3x^2 + 1$.

Completing the Square

If $x, y \in \mathbb{R}$ we have $\boxed{(x \pm y)^2 = x^2 \pm 2xy + y^2}$.

In particular, $\left(x \pm \frac{1}{2}b\right)^2 = x^2 \pm bx + \left(\frac{b}{2}\right)^2$.

Therefore, $x^2 \pm bx = \left(x \pm \frac{1}{2}b\right)^2 - \frac{1}{4}b^2$.

Consider $x^2 - x + 1$.

$x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$.

So $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$.

Question 3

What is the range of $f(x) = x^2 - x + 1$?

Quadratic Functions: Roots

A **root** of a function f is a point x such that $f(x) = 0$.

How to find the roots of a quadratic? We can complete the square. For example, to solve

$$x^2 - 4x + 1 = 0$$

we complete the square first: $x^2 - 4x + 1 = (x-2)^2 - 4 + 1 = (x-2)^2 - 3$.

We want $(x-2)^2 - 3 = 0$.

This is $(x-2)^2 = 3$.

Two options: $x - 2 = \sqrt{3}$ or $x - 2 = -\sqrt{3}$.

So, two roots: $\boldsymbol{x = 2 + \sqrt{3}}$ and $\boldsymbol{x = 2 - \sqrt{3}}$.

Question 4

What about $f(x) = x^2 - x + 1$?

Quadratic Functions: Roots

There is a formula: if $f(x) = ax^2 + bx + c$ with $a \neq 0$ the roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This only works if $b^2 - 4ac \geqslant 0$.

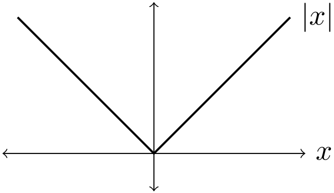
If $b^2 - 4ac < 0$ then the quadratic doesn't have roots.

Question 5
Use the formula for $x^2 - x + 1 = 0$ and $x^2 - 4x + 1 = 0$.

Absolute Value

We define the **absolute value** function $|\cdot| : \mathbb{R} \to [0, +\infty)$ by

$$|\boldsymbol{x}| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0. \end{cases}$$



Example: $|2| = 2$, $|0| = 0$, but $|-1| = 1$.

We have $|xy| = |x||y|$ and $|x + y| \leqslant |x| + |y|$ for all $x, y \in \mathbb{R}$.

Inverse Functions

Let $f : X \to Y$ be a function. Suppose that

$$\text{for each } y \in Y \text{ there is exactly one } x \in X \text{ such that } f(x) = y.$$

In that case we say that f is **invertible** or a **bijection**.

We obtain a function $\boldsymbol{f^{-1}} : Y \to X$ such that $f(f^{-1}(y)) = y$ for all $y \in Y$. We call it the **inverse** function of f.

Inverse Functions

"f^{-1} is the inverse of f" is equivalent to

$$\text{for every } x \in X,\ y \in Y,\ f(x) = y \text{ if and only if } f^{-1}(y) = x.$$

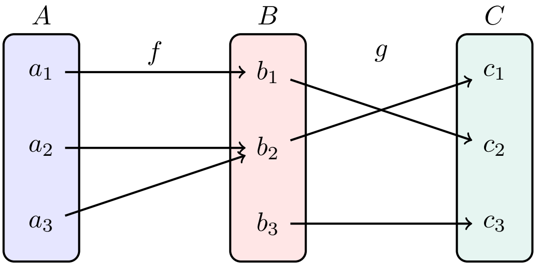
Notice the symmetry: if f^{-1} is the inverse of f then f is the inverse of f^{-1}.

Examples.

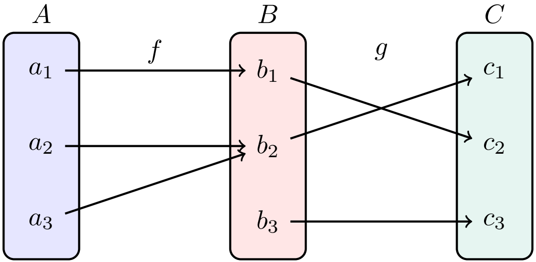
1. If $f(x) = x + 1$ then $f^{-1}(x) = x - 1$, since $x + 1 = y$ if and only if $x = y - 1$.
2. The inverse of $f(x) = x^{\frac{1}{2}}$ is $g(x) = x^2$ defined only for $x \geqslant 0$.
3. log is the inverse of exp.
4. $\log_a(x)$ is the inverse of $f(x) = a^x$ if $a > 0$, $a \neq 1$.
5. The function $f(x) = 1/x$ is its own inverse.
6. The absolute value $|\cdot|$ doesn't have an inverse.

Question 6

Do either f or g have an inverse? If so, find it.



Answer



f

x	y
a_1	b_1
a_2	b_2
a_3	b_2

f^{-1} doesn't exist

x	y
b_1	a_1
b_2	a_2 or a_3?
b_3	?

g

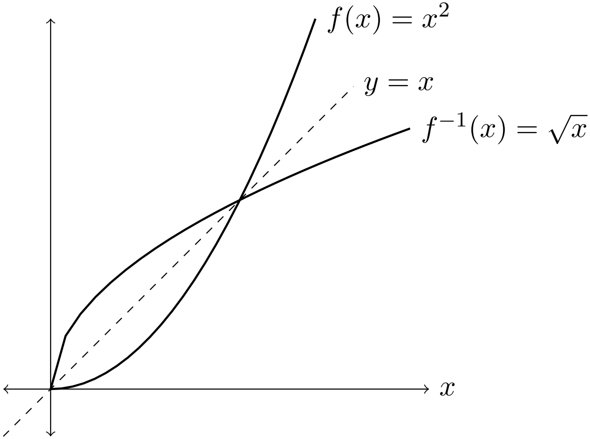
x	y
b_1	c_1
b_2	c_2
b_3	c_3

g^{-1}

x	y
c_1	b_2
c_2	b_1
c_3	b_3

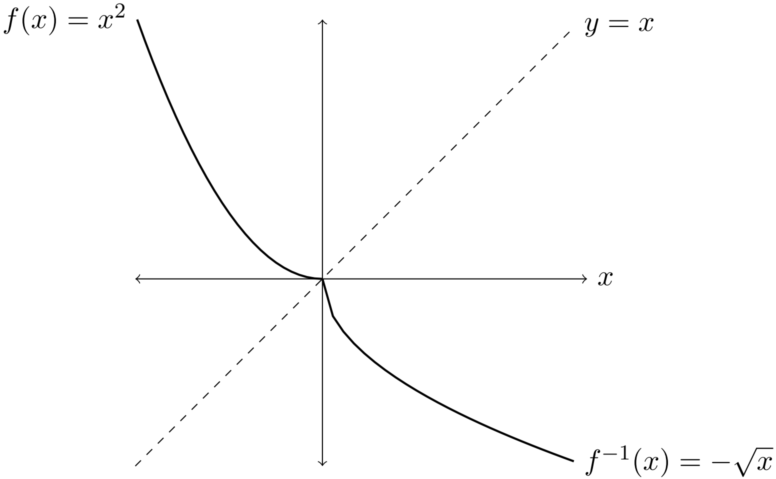
Inverses as Reflections About the 45° Line

$$f : [0, +\infty) \to [0, +\infty),\ f(x) = x^2$$



Inverses as Reflections About the 45° Line

$$f : (-\infty, 0] \to [0, +\infty),\ f(x) = x^2$$



Question 7

Does f have an inverse? If so, find it.

1. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 1$,
2. $f : \mathbb{R} \to [1, +\infty)$ given by $f(x) = x^2 + 1$,
3. $f : [0, +\infty) \to [1, +\infty)$ given by $f(x) = x^2 + 1$,
4. $f : (-\infty, 0] \to [1, +\infty)$ given by $f(x) = x^2 + 1$.