Lecture 1: Basic Algebra

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Plan

Number sets

1. Natural Numbers
2. Integers
3. Rationals
4. Reals

Operations on numbers

1. Sum, Subtraction, Multiplication, Division
2. Exponentiation
3. Exp and Log

Potato and Kiki

Numbers

Natural numbers. Denoted by $\mathbb{N} = \{1, 2, \ldots\}$. Sometimes people include 0.

Integers. Denoted by $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Rational numbers. These are fractions like $-\frac{31}{13}$ and 2. Denoted by

$$\mathbb{Q} = \left\{ \frac{n}{m} : n, m \in \mathbb{Z}, m \neq 0 \right\}.$$

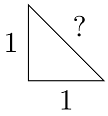
We can write rational numbers as

$$n.d_1 \ldots d_k \overline{d_{k+1} \ldots d_l} = n.d_1 \ldots d_k \underbrace{d_{k+1} \ldots d_l}\underbrace{d_{k+1} \ldots d_l} \ldots,$$

with $n \in \mathbb{Z}$ and digits $d_1, \ldots, d_l \in \{0, \ldots, 9\}$. E.g., $\frac{1}{2} = 0.5 = 0.4\overline{9}$ and $\frac{10511}{4950} = 2.12\overline{34}$.

Real Numbers

The set of real numbers $\mathbb{R}$ is the "completion" of $\mathbb{Q}$ in some sense. Why do we want this?



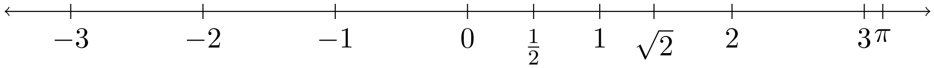
The length of the long side of that triangle is $\sqrt{2}$, which is not a rational number.

We can write real numbers as

$$n.d_1 d_2 \ldots$$

for any $n \in \mathbb{Z}$ and any infinite list of digits $d_1, d_2, \ldots \in \{0, \ldots, 9\}$.

We can visualize real numbers as points in an infinite line:



Operations

Let's talk about the operations $+, -, \times, \div$ (division) and $\bullet^{\bullet}$ (exponentiation).

For any $x, y \in \mathbb{R}$ we can obtain $\boldsymbol{x + y}$, $\boldsymbol{x - y}$, $\boldsymbol{x \times y}$ (also denoted by $\boldsymbol{xy}$ or $\boldsymbol{x \cdot y}$), and, if $y \neq 0$, also $\boldsymbol{\frac{x}{y}}$ (also denoted by $\boldsymbol{x/y}$).

We can perform these operations within $\mathbb{Q}$ if $x, y \in \mathbb{Q}$.

But in general we cannot divide within $\mathbb{Z}$, e.g., $\frac{1}{2} \notin \mathbb{Z}$. And we cannot subtract arbitrary $x, y \in \mathbb{N}$ within $\mathbb{N}$, e.g., $1 - 2 \notin \mathbb{N}$.

We can think of $\mathbb{Z}$ as the minimal extension of $\mathbb{N}$ in which we can subtract, and $\mathbb{Q}$ as the minimal extension of $\mathbb{Z}$ in which we can also divide by nonzero numbers.

Exponentiation

For $x \neq 0$ we define $\boldsymbol{x^0} = 1$ and $\boldsymbol{x^n} = \underbrace{x \times \cdots \times x}_{n \text{ times}}$ for any $n \in \mathbb{N}$. We define $\boldsymbol{x^{-n}} = 1/x^n$.

Question 1
Convince yourself that $x^{n+m} = x^n \times x^m$ and $(x^n)^m = x^{n \times m}$ for any $x \neq 0$ and $n, m \in \mathbb{Z}$.

Question 2
Convince yourself that $(xy)^n = x^n y^n$ and $(x/y)^n = x^n/y^n$ for any $x, y \neq 0$ and $n \in \mathbb{Z}$.

Exponentiation

If $x > 0$ and $n \in \mathbb{N}$ we define $\boldsymbol{x^{\frac{1}{n}}}$ (or $\boldsymbol{\sqrt[n]{x}}$) as the number $y > 0$ that satisfies $y^n = x$.

If $x > 0$ and $q \in \mathbb{Q}$ with $q = \frac{n}{m}$, $n \in \mathbb{Z}, m \in \mathbb{N}$ we define $\boldsymbol{x^q}$ as $(x^n)^{\frac{1}{m}}$.

For example, $4^{\frac{1}{2}} = \sqrt{4} = 2$, since $4 = 2^2$, and $(\frac{1}{125})^{-\frac{1}{3}} = 5$ since $5^{-3} = \frac{1}{125}$.

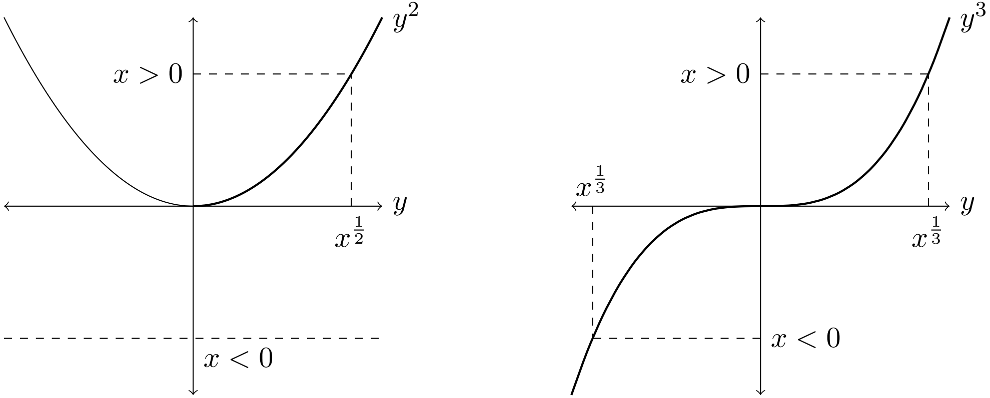
If $x > 0$ and $r \in \mathbb{R}$ we can define $\boldsymbol{x^r}$ as the number that can be approximated by x^q if we take $q \in \mathbb{Q}$ very close to r. (We will make this precise later.)

We can define $0^r = 0$ if $r > 0$. (Why don't we want to define 0^r for $r \leqslant 0$?)

What about $x < 0$? There is no $y \in \mathbb{R}$ such that $y^2 = -1$, so we can't define $(-1)^{\frac{1}{2}}$ in $\mathbb{R}$. But we can define $x^{\frac{1}{3}}$ for any $x \in \mathbb{R}$ as the number y such that $y^3 = x$, e.g., $(-1)^{\frac{1}{3}} = -1$.

Question 3
Why?

$x^{\frac{1}{2}}$ AND $x^{\frac{1}{3}}$



Indexed Sums

If we have a list of numbers $x_1, \ldots, x_n$ we denote their sum by

$$\sum_{i=1}^{n} x_i = x_1 + \cdots + x_n.$$

In general $\sum_{i=a}^{b}$ expression is the sum of *expression* with i replaced by a, then by $a + 1$, etc, until i is replaced by b.

We can also write $\sum_{x \in A}$ expression, which is the sum of *expression* with x replaced by each member of the set A.

Example:

$$\sum_{\substack{n \in \mathbb{N} \\ n \leqslant 6 \\ n \text{ odd}}} (n+1)^2 = (1+1)^2 + (3+1)^2 + (5+1)^2.$$

Indexed Products

If we have a list of numbers $x_1, \ldots, x_n$ we denote their product by

$$\prod_{i=1}^{n} x_i = x_1 \times \cdots \times x_n.$$

Question 4

Convince yourself that

1. $\sum_{i=1}^{n} x = nx$,
2. $\prod_{i=1}^{n} x = x^n$,
3. $\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$,
4. $\prod_{i=1}^{n} \frac{x_i}{y_i} = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} y_i}$ if $y_i \neq 0$.

Order of Operations

We evaluate exponentiations first, then divisions and multiplications, then subtractions and sums.

Example: if $x = -1$, $n = 1$,

$$\begin{aligned} 1 + 2 \times x^3 - 3 \times \frac{4^n}{2} &= 1 + 2 \times (-1)^3 - 3 \times \frac{4^1}{2} \\ &= 1 + 2 \times (-1) - 3 \times \frac{4}{2} \\ &= 1 + (-2) - 6 \\ &= -7. \end{aligned}$$

Parentheses

We use parentheses to break this order. We evaluate first what's in the parentheses. Example: if $x = -1$, $n = 1$,

$$\begin{aligned}(1+2) \times (x^3 - 3) \times \frac{4^n}{2} &= (1+2) \times ((-1)^3 - 3) \times \frac{4^1}{2} \\ &= 3 \times ((-1) - 3) \times 2 \\ &= 3 \times (-4) \times 2 \\ &= -24.\end{aligned}$$

People sometimes use brackets $[\ldots]$ and curly braces $\{\ldots\}$ as parentheses. For example,

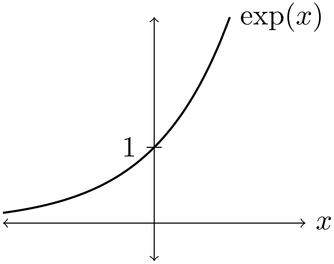
$$\sum_{i=1}^{n} \left[-\frac{1}{2\sigma} \left(y_i - \sum_{k=1}^{K} \beta_k x_{ik} \right)^2 \right] + \left\{ -\lambda \left[\sum_{k=1}^{K} (\beta_k)^2 + \sum_{k=1}^{K} |\beta_k| \right] \right\}$$

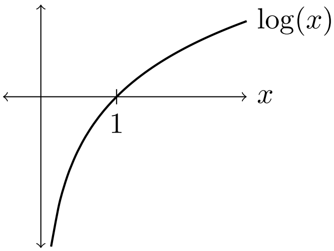
Exp and Log

Euler's constant is the irrational number $\boldsymbol{e} = 2.71828\ldots$ defined as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

We define $\mathbf{exp}(\boldsymbol{x}) = e^x$ for any $x \in \mathbb{R}$, and $\mathbf{log}(\boldsymbol{x})$ for any $x > 0$ as the number y such that $\exp(y) = x$. Some people call "natural logarithm" what I call log, and write **ln** instead. Plot:





Exp and Log

Question 5

Convince yourself that for any $x, y, a, b \in \mathbb{R}$ such that $a, b > 0$,

1. $e^{\log(a)} = a$,
2. $\log(e^x) = x$,
3. $\exp(x + y) = \exp(x)\exp(y)$,
4. $\log(ab) = \log(a) + \log(b)$,
5. $\exp(xy) = \exp(x)^y$,
6. $\log(a^x) = x\log(a)$.

If $a, x > 0$ and $a \neq 1$ we define $\boldsymbol{\log_a(x)}$ as the number y such that $a^y = x$.

Question 6

Convince yourself that for every $a, x \in \mathbb{R}$ such that $a, x > 0$ and $a \neq 1$ we have

$$\log_a(x) = \frac{\log(x)}{\log(a)}.$$

Simplify Expressions

Question 7

Simplify the following expressions:

1. $\dfrac{2x/6}{4/3x}$
2. $\dfrac{2^x 4^y}{\sqrt{4^{x+y} 8^y}}$
3. $\log(2) + \log(1/2)$
4. $\log_5(125)$
5. $\log\left\{\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)\right]\right\}$

Lunch Break