Section 7

Repeated elections with two parties

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Plan

We are going to discuss the following paper:

- Bernhardt, Dan, Larissa Campuzano, Francesco Squintani, and Odilon Câmara. 2009. "On the Benefits of Party Competition." *Games and Economic Behavior* 66(2): 685–707.

It is the same model as Duggan (2000), only adding two parties, A and B.

- A is the party of left-of-center candidates, i.e., $A = \{x : x < 0\}$;
- B is the party of right-wing candidates, i.e., $B = \{x : x > 0\}$.
- In each election, if the incumbent is of party A (say), then the challenger is a randomly selected member of party B (instead of from the whole population as in Duggan).

The main result:

- *Every* voter is better off having parties!

The model

- At each time k an incumbent chooses a policy $x_k \in I = [-1, 1]$.
- Citizens have types symetrically distributed with density f over I.
- Per-period utility from policy x for type t is $u_t(x) = -|x - t|$.
- There are two parties: $A = [-1, 0)$ and $B = [0, 1]$.
  - They alternate in power unless the incumbent is reelected.
- In each period there is an election; if the incumbent gets $\geq \frac{1}{2}$ of the votes, she keeps office.
  - The challenger is selected at random from the opposition party.
- Every citizen's discount factor is $\delta \in (0, 1)$.

Equilibrium

We look for a stationary PBE with the following characteristics:

- I is partitioned into $W = [-w, w]$, $C = [-c, c] \setminus W$, $E = I \setminus C$, with $0 < w < c < 1$.
- Each candidate of type t chooses policy $x = p_t$, where

$$p_t = \begin{cases} t, & \text{if } t \in W \cup E, \\ -w, & \text{if } t \in [-c, -w), \text{ and} \\ w, & \text{if } t \in (w, c] \end{cases}$$

  if she is in office.
- If the voters see an off-equilibrium policy x, i.e., $x \in C$, then they assume that $t = x$.
- The incumbent wins iff $x \in W$.

Let's check that these strategies are part of a PBE for some w, c.

The voters' decision

Take the strategies of candidates as given.

Let x be the type of the incumbent, and let $\bar{x}$ be the type of a voter.

Let's assume that $x < 0$, i.e., $x \in A$.

The utility from retaining x is $u_{\bar{x}}(p_x)$.

The utility from choosing a random challenger from B is

$$
\begin{aligned}
U_{\bar{x}}^{B} = \Pr(t \notin E \mid t \in B)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in B \setminus E] + \\
+ \Pr(t \in E \mid t \in B)\left\{(1-\delta)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in B \cap E] + \delta U_{\bar{x}}^{A}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
U_{\bar{x}}^{A} = \Pr(t \notin E \mid t \in A)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in A \setminus E] + \\
+ \Pr(t \in E \mid t \in A)\left\{(1-\delta)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in A \cap E] + \delta U_{\bar{x}}^{B}\right\}.
\end{aligned}
$$

So, $\bar{x}$ votes to retain $x \in A$ iff $u_{\bar{x}}(x) \geq U_{\bar{x}}^{B}$.

Similarly, $\bar{x}$ votes to retain $x \in B$ iff $u_{\bar{x}}(x) \geq U_{\bar{x}}^{A}$.

We can do the same that we did in lecture: prove that the median voter decides by first bounding

$$\frac{\partial U_{\bar{x}}^{A}}{\partial \bar{x}} \text{ and } \frac{\partial U_{\bar{x}}^{B}}{\partial \bar{x}}$$

where $U_{\bar{x}}^{A}, U_{\bar{x}}^{B}$ are differentiable. (They are for all but finitely many points.)

We have

$$
\begin{aligned}
U_{\bar{x}}^{B} &= \Pr(t \notin E \mid t \in B)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in B \setminus E] + \\
&\quad + \Pr(t \in E \mid t \in B)\left\{(1-\delta)\mathbb{E}[u_{\bar{x}}(p_t) \mid t \in B \cap E] + \delta U_{\bar{x}}^{A}\right\} = \\
&= \Pr(t \in W \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in W \cap B] + \\
&\quad + \Pr(t \in C \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in \{-w, w\} \cap B] + \\
&\quad + \Pr(t \in E \mid t \in B)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) \mid t \in E \cap B] + \\
&\quad + \beta U_{\bar{x}}^{A} = \\
&= \Pr(t \in W \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in W \cap B] + \\
&\quad + \Pr(t \in C \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in \{-w, w\} \cap B] + \\
&\quad + \Pr(t \in E \mid t \in B)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) \mid t \in E \cap B] + \\
&\quad + \beta\Pr(t \in W \mid t \in A)\mathbb{E}[u_{\bar{x}}(t) \mid t \in W \cap A] + \\
&\quad + \beta\Pr(t \in C \mid t \in A)\mathbb{E}[u_{\bar{x}}(t) \mid t \in \{-w, w\} \cap A] + \\
&\quad + \beta\Pr(t \in E \mid t \in A)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) \mid t \in E \cap A] + \\
&\quad + \beta^2 U_{\bar{x}}^{B},
\end{aligned}
$$

where $\beta = \Pr(t \in E \mid t \in B)\delta = \Pr(t \in E \mid t \in A)\delta$.

$$
\begin{aligned}
U_{\bar{x}}^{B} &= \Pr(t \in W \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in W \cap B] + \\
&\quad + \Pr(t \in C \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) \mid t \in \{-w, w\} \cap B] + \\
&\quad + \Pr(t \in E \mid t \in B)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) \mid t \in E \cap B] + \\
&\quad + \Pr(t \in W \mid t \in B)\mathbb{E}[\beta u_{\bar{x}}(-t) \mid t \in W \cap B] + \\
&\quad + \Pr(t \in C \mid t \in B)\mathbb{E}[\beta u_{\bar{x}}(-t) \mid t \in \{-w, w\} \cap B] + \\
&\quad + \Pr(t \in E \mid t \in B)(1-\delta)\mathbb{E}[\beta u_{\bar{x}}(-t) \mid t \in E \cap B] + \\
&\quad + \beta^2 U_{\bar{x}}^{B} = \\
&= \Pr(t \in W \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in W \cap B] + \\
&\quad + \Pr(t \in C \mid t \in B)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in \{-w, w\} \cap B] + \\
&\quad + \Pr(t \in E \mid t \in B)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in E \cap B] + \\
&\quad + \beta^2 U_{\bar{x}}^{B}.
\end{aligned}
$$

After solving for $U_{\bar{x}}^B$ we can re-write this as

$$U_{\bar{x}}^B = \sum_{i=1}^{3} \omega_i \mathbb{E}\left[\frac{u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t)}{1+\beta} \mid t \in S_i\right]$$

for some $\omega_1, \omega_2, \omega_3 \in [0,1]$ s.t. $\sum_{i=1}^{3} \omega_i = 1$, and $S_i \subset B$.

Now, for $t \geq 0$,

$$\frac{\partial}{\partial \bar{x}} \frac{u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t)}{1+\beta} = \begin{cases} 1, & \text{if } \bar{x} < -t, \\ \frac{1-\beta}{1+\beta} \in (0,1), & \text{if } \bar{x} \in (-t, t), \\ -1, & \text{if } \bar{x} > t. \end{cases}$$

So, if $\bar{x} \leq 0$, we have $0 \leq \frac{\partial}{\partial \bar{x}} \frac{u_{\bar{x}}(t)+\beta u_{\bar{x}}(-t)}{1+\beta} \leq 1$, and if $\bar{x} > 0$, we have $-1 \leq \frac{\partial}{\partial \bar{x}} \frac{u_{\bar{x}}(t)+\beta u_{\bar{x}}(-t)}{1+\beta} \leq 1$.

Applying Leibniz Rule, we get that if $\bar{x} \leq 0$ then $0 \leq \frac{\partial U_{\bar{x}}^B}{\partial \bar{x}} \leq 1$, and if $\bar{x} > 0$ then $-1 \leq \frac{\partial U_{\bar{x}}^B}{\partial \bar{x}} \leq 1$.

We can do the same thing for $U_{\bar{x}}^A$:

$$\begin{aligned} U_{\bar{x}}^A = \frac{1}{1-\beta^2}\Big\{ &\Pr(t \in W \mid t \in A)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in W \cap A] + \\ &+ \Pr(t \in C \mid t \in A)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in \{-w, w\} \cap A] + \\ &+ \Pr(t \in E \mid t \in A)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) + \beta u_{\bar{x}}(-t) \mid t \in E \cap A]\Big\}. \end{aligned}$$

where $\beta = \Pr(t \in E \mid t \in A)\delta$.

We get that if $\bar{x} < 0$ then $-1 \leq \frac{\partial U_{\bar{x}}^A}{\partial \bar{x}} \leq 1$, and if $\bar{x} \geq 0$ then $-1 \leq \frac{\partial U_{\bar{x}}^A}{\partial \bar{x}} \leq 0$.

Also, note that $u_{\bar{x}}(t)$ is concave in $\bar{x}$ for any t (positive or negative), so $U_{\bar{x}}^A$ and $U_{\bar{x}}^B$ are concave.

The median voter is decisive

Suppose that $t < 0$ and $u_0(t) - U_0^B > 0$, i.e., the median type votes to reelect the incumbent. We will show that every $\bar{x} < 0$ does as well.

For $t \leq \bar{x} < 0$, we have $u_{\bar{x}}(t) \geq u_0(t)$ and $U_{\bar{x}}^B \leq U_0^B$ (since we proved that $\frac{\partial U_{\bar{x}}^B}{\partial \bar{x}} \geq 0$ for $\bar{x} < 0$), so $u_{\bar{x}}(t) - U_{\bar{x}}^B \geq u_0(t) - U_0^B > 0$, as desired.

For $\bar{x} < t$, note that $u_{\bar{x}}(t) = -(t - \bar{x})$ is linear in $\bar{x}$, so $u_{\bar{x}}(t) - U_{\bar{x}}^B$ is convex, hence to prove that $u_{\bar{x}}(t) - U_{\bar{x}}^B > 0$ for all $\bar{x} \in [-1, t]$ it's enough to prove it for both endpoints.

We already did $\bar{x} = t$, so we just need to check that $u_{-1}(t) - U_{-1}^B > 0$. Now, looking back at the formula for $U_{\bar{x}}^B$ and plugging $\bar{x} = -1$ we can see that $U_{-1}^B < -1$, but $u_{-1}(t) = -(t + 1) \geq -1$, so $u_{-1}(t) - U_{-1}^B > 0$, as desired.

Suppose that $t < 0$ and $u_0(t) - U_0^B < 0$, i.e., the median type votes for the opposition. We will show that every $\bar{x} > 0$ does as well.

We have

$$U_{\bar{x}}^B - U_0^B = \int_0^{\bar{x}} \frac{\partial U_x^B}{\partial x} dx \geq \int_0^{\bar{x}} (-1) dx = -\bar{x},$$

and $u_{\bar{x}}(t) - u_0(t) = -\bar{x}$, so

$$u_{\bar{x}}(t) - U_{\bar{x}}^B \geq u_0(t) - \bar{x} - (U_0^B - \bar{x}) > 0,$$

as desired.

Proving that the median type is decisive for $t > 0$ is similar.

The median voter's decision

An incumbent choosing policy $t < 0$ is reelected iff $u_0(p_t) \geq U_0^B$, and $t > 0$ iff $u_0(p_t) \geq U_0^A$.

We have

$$\begin{aligned}
U_0^A = \frac{1}{1-\beta^2}\Big\{ &\Pr(t \in W \mid t \in A)\mathbb{E}[u_0(t) + \beta u_0(-t) \mid t \in W \cap A] + \\
&+ \Pr(t \in C \mid t \in A)\mathbb{E}[u_0(t) + \beta u_0(-t) \mid t \in \{-w, w\} \cap A] + \\
&+ \Pr(t \in E \mid t \in A)(1-\delta)\mathbb{E}[u_0(t) + \beta u_0(-t) \mid t \in E \cap A]\Big\} \\
= \frac{-1}{1-\beta}\Big\{ &\Pr(t \in W \mid t \in A)\mathbb{E}[t \mid t \in [0, w]] + \\
&+ \Pr(t \in C \mid t \in A)w + \Pr(t \in E \mid t \in A)(1-\delta)\mathbb{E}[t \mid t \in (c, 1]]\Big\},
\end{aligned}$$

and $U_0^B = U_0^A$.

Hence, t is reelected iff $-|p_t| \geq U_0^A$.

This implies that $U_0^A = -w$, because if $U_0^A < -w$ then $t > w$ would choose $p_t = t$ instead of $p_t = w$, and if $U_0^A > -w$ then $t = w$ would not win when choosing $p_t = w$.

We have our first equilibrium condition: $\boxed{U_0^A = -w}$.

The incumbent's decision

If $t \in W = [-w, w]$, she can choose $x = t$ and win, so she will.

If $t \in (w, c]$, we need that $u_t(w) \geq (1-\delta)u_t(t) + \delta U_t^A$, i.e., $u_t(w) \geq \delta U_t^A$.

If $t \in (c, 1]$, we need that $(1-\delta)u_t(t) + \delta U_t^A \geq u_t(w)$, i.e., $u_t(w) \leq \delta U_t^A$.

Hence, at $t = c$ we must have equality: $u_c(w) = \delta U_c^A$.

We have our second equilibrium condition: $\boxed{\delta U_c^A = -(c-w)}$.

It can be proved (see the paper) that these two equations indeed have a solution and that it is unique.

Back to the model without parties

If there are no parties, we have

$$\begin{aligned} U_{\bar{x}} = \frac{1}{1-\beta}\Big\{ &\Pr(t \in W)\mathbb{E}[u_{\bar{x}}(t) \mid t \in W] + \\ &+ \Pr(t \in C)\mathbb{E}[u_{\bar{x}}(t) \mid t \in \{-w, w\}] + \\ &+ \Pr(t \in E)(1-\delta)\mathbb{E}[u_{\bar{x}}(t) \mid t \in E]\Big\}. \end{aligned}$$

where $\beta = \Pr(t \in E)\delta$.

We can do the same analysis, and we arrive at the same two conditions for an equilibrium:

$$\begin{aligned} U_0 &= -w \\ \delta U_c &= -(c-w). \end{aligned}$$

Next steps

We won't prove these, but the next steps in the paper are:

- We note that $U_0(w,c) = U_0^A(w,c)$, so $U_0(w,c) = -w$ holds with and without parties. Using that we get that if w increases, c decreases.
- We note that $U_{\bar{x}}^A(w,c) > U_{\bar{x}}(w,c) > U_{\bar{x}}^B(w,c)$ for negative $\bar{x}$.
- We prove that if w, c is the equilibrium without parties and $\tilde{w}, \tilde{c}$ is the equilibrium with parties, then $0 < \tilde{w} < w < c < \tilde{c} < 1$.
- We verify that every voter is (ex ante) better off with parties than without.