Section 11

Monotone comparative statics and economic policy

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Plan for today

- Talk about monotone comparative statics
- Talk about variations on Meltzer-Richard

Increasing differences

We say that the function $f(x, y)$ has (weak) *increasing differences* if for every $x > x'$, $y > y'$, we have

$$f(x, y) - f(x', y) \geq f(x, y') - f(x', y').$$

In other words, $f(x, \cdot) - f(x', \cdot)$ is (weakly) increasing when $x > x'$.

If f is C^2 (i.e., twice differentiable with continuous partial derivatives) then this condition is equivalent to

$$\frac{\partial^2 f}{\partial x \partial y} \geq 0$$

everywhere.

Equivalence

Why?

Suppose that $\frac{\partial^2 f}{\partial x \partial y} \geq 0$. Let $x > x'$ and $g(y) = f(x, y) - f(x', y)$. Increasing differences requires that $g(y)$ is weakly increasing, i.e., $g'(y) \geq 0$, i.e., $\frac{\partial f}{\partial y}(x, y) - \frac{\partial f}{\partial y}(x', y) \geq 0$, i.e., $\frac{\partial f}{\partial y}(x, y) \geq \frac{\partial f}{\partial y}(x', y)$, i.e., $\frac{\partial f}{\partial y}(\cdot, y)$ is increasing. Now, $\frac{\partial f}{\partial y}(\cdot, y)$ is increasing iff $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ everywhere, which is true. Hence, we have increasing differences.

Suppose that $\frac{\partial^2 f}{\partial x \partial y} < 0$ for some x, y. By continuity, this holds in an open neighborhood of (x, y). By the previous argument, in that set we have that f has strictly *decreasing* differences. Hence it doesn't have increasing differences.

Supermodularity

We say that the function $f : \mathbb{R}^n \to \mathbb{R}$ is *supermodular* if for every $x, y \in \mathbb{R}^n$ we have

$$f(\max(x, y) + f(\min(x, y)) \geq f(x) + f(y),$$

where $\max(x, y) := (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$ and $\min(x, y) := (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$.

A *sublattice* of $\mathbb{R}^n$ is a subset S such that if $x, y \in S$, we have $\max(x, y), \min(x, y) \in S$. For example, $S = I_1 \times \cdots \times I_n$ where I_i are intervals. We can extend the definition to functions $f : S \to \mathbb{R}$ where S is a sublattice.

Proposition. f is supermodular iff $f(x_1, \ldots, x_n)$ has increasing differences in x_i, x_j for every $i \neq j$.

Proof. One implication (supermodularity implies increasing differences) is easy

I'll do the other for $n = 3$. It generalizes easily.

Given $(x,y,z),(x',y',z')$ we want to show that

$$f(\max\{x,x'\},\max\{y,y'\},\max\{z,z'\})+f(\min\{x,x'\},\min\{y,y'\},\min\{z,z'\})\geq f(x,y,z)+f(x',y',z').$$

Indeed, we have

$$\begin{aligned}
&f(\max\{x,x'\},\max\{y,y'\},\max\{z,z'\})-f(x,y,z)\\
&=f(\max\{x,x'\},\max\{y,y'\},\max\{z,z'\})-f(\max\{x,x'\},\max\{y,y'\},z)\\
&\quad+f(\max\{x,x'\},\max\{y,y'\},z)-f(\max\{x,x'\},y,z)\\
&\quad+f(\max\{x,x'\},y,z)-f(x,y,z)\\
&=f(\max\{x,x'\},\max\{y,y'\},z')-f(\max\{x,x'\},\max\{y,y'\},\min\{z,z'\})\\
&\quad+f(\max\{x,x'\},y',z)-f(\max\{x,x'\},\min\{y,y'\},z)\\
&\quad+f(x',y,z)-f(\min\{x,x'\},y,z)\\
&\geq f(x',y',z')-f(x',y',\min\{z,z'\})\\
&\quad+f(x',y',\min\{z,z'\})-f(x',\min\{y,y'\},\min\{z,z'\})\\
&\quad+f(x',\min\{y,y'\},\min\{z,z'\})-f(\min\{x,x'\},\min\{y,y'\},\min\{z,z'\})\\
&=f(x',y',z')-f(\min\{x,x'\},\min\{y,y'\},\min\{z,z'\}),
\end{aligned}$$

as desired. ■

Corolary. If f is C^2 then it is supermodular iff

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$$

for every pair of variables x_i, x_j, $i \neq j$.

Let $A, A' \subset \mathbb{R}^n$ be sublattices. We say that $A \geq A'$ iff for every $x \in A, x' \in A'$ we have $\max(x, x') \in A$ and $\min(x, x') \in A$.

Let $f(x,t) : S \times T \to \mathbb{R}$ be a function and let $f^*(t) = \operatorname*{argmax}_{x \in S} f(x,t)$. Assume that it is not empty for every $t \in T$. This is the case if, e.g., f is continuous and S is compact (i.e., closed and bounded).

Topkis Theorem

Theorem. If f is supermodular then $f^*(t)$ is a sublattice and $f^*(t) \geq f^*(t')$ if $t \geq t'$ component-wise (i.e., $t_i \geq t'_i$ for every i).

Proof. If $x, x' \in f^*(t)$,

$$f(\max\{x,x'\},t) + f(\min\{x,x'\},t) \geq f(x,t) + f(x',t) \geq f(\max\{x,x'\},t) + f(\min\{x,x'\},t),$$

so $f(\max\{x,x'\},t) = f(\min\{x,x'\},t) = f(x,t)$ and $\max\{x,x'\}, \min\{x,x'\} \in f^*(t)$, as claimed.

If $t \geq t'$, $x \in f^*(t)$, $x' \in f^*(t')$, we have

$$f(\max\{x,x'\},\max\{t,t'\}) + f(\min\{x,x'\},\min\{t,t'\}) \geq f(x,t) + f(x',t') \geq f(\max\{x,x'\},t) + f(\min\{x,x'\},t'),$$

so $f(\max\{x,x'\},t) = f(x,t)$ and $f(\min\{x,x'\},t') = f(x',t')$, hence $\max\{x,x'\} \in f^*(t)$, $\min\{x,x'\} \in f^*(t')$, and thus $f^*(t) \geq f^*(t')$, as claimed. ■

People often use the following corolary: there is a minimum $x_*(t)$ and a maximum $x^*(t)$ in $f^*(t)$ and $x_*(t)$, $x^*(t)$ are weakly increasing functions of t.

Strategic complementarities

Consider a game with players $1, \ldots, N$. Player i chooses $x_i \in S_i$ where S_i is a sublattice of $\mathbb{R}^{n_i}$. They have utilities $u_i(x_1, \ldots, x_n, t)$, where t is an exogenous parameter.

If $u_i(x, t)$ has increasing differences in x_i, x_j then we say that x_i, x_j are *strategic complements* for i. In words, a "higher" action by j induces i to choose a higher action.

(If u_i has decreasing differences, then x_i, x_j are strategic *substitutes*. In that case, the more j does, the less i wants to do.)

Proposition. Suppose that $u_i(x, t)$ is supermodular in x, u_i is continuous and S_i is compact for every i. Then there is a Nash equilibrium in pure strategies.

Proof. (Sketch.) Take the smallest action profile x. For each i take x_i' to be the largest best response by i to x_{-i}. Iterate. By Topkis theorem, the sequence of action profiles is increasing. By compactness, it converges. By continuity, the limit is an equilibrium. ■

I won't prove it, but there is a maximum and a minimum PSNE, and they are weakly increasing in the parameter t if $u_i(x, t)$ are supermodular (Milgrom & Shannon 1994).

Supermodularity implies *monotone comparative statics* (equilibrium x is increasing in t).

Single-crossing property

Take $u(x,t)$. Think of x as a policy and t as a type. We have that u is single-crossing iff, for every $x' > x$, we have that

- $u(x',t) > u(x,t)$ implies that $u(x',t') > u(x,t')$ for all $t' > t$ and
- $u(x,t) > u(x',t)$ implies that $u(x,t') > u(x',t')$ for all $t' < t$.

Proposition. Let $u(x,t)$ be single-crossing. If t^* is the median type and x^* is her strictly preferred policy then x^* is a Condorcet winner.

Proof. Let x be other policy.

- If $x < x^*$ then $u(x^*,t^*) > u(x,t^*)$ implies $u(x^*,t) > u(x,t)$ for every $t > t^*$, so a majority prefers x^* to x.
- If $x > x^*$ then $u(x^*,t^*) > u(x,t^*)$ implies $u(x^*,t) > u(x,t)$ for every $t < t^*$, so again a majority prefers x^* to x. ■

Supermodularity implies single crossing

Proposition. If $u(x,t)$ is supermodular then it is single crossing.

Proof. We have $u(x',t') - u(x,t') \geq u(x',t) - u(x,t)$ for $x' > x$, $t' > t$, so

- if $u(x',t) > u(x,t)$ then $u(x',t) - u(x,t) > 0$, $u(x',t') - u(x,t') > 0$ and $u(x',t') > u(x,t')$ for $t' > t$, and
- if $u(x,t') > u(x',t')$ then $0 > u(x',t') - u(x,t')$, $0 > u(x',t) - u(x,t)$ and $u(x,t) > u(x',t)$ for $t < t'$. ∎

So, to check that $u(x,t)$ is single-crossing, it is enough to verify that

$$\frac{\partial^2 f}{\partial x \partial y} \geq 0$$

everywhere.

Example: unemployment insurance

Individuals have employment with probability p, in which case they receive income x. Otherwise, they are unemployed and receive a benefit b that is the same for everyone.

b is financed by a linear income tax τ over the employed. There is no distortion. There is budget balance:

$$\tau \int px\phi(p,x)\,dpdx = \int (1-p)b\phi(p,x)\,dpdx,$$

where $\phi(p,x)$ is the density of voters with p, x. Hence we have

$$b = \tau \frac{\overline{px}}{1-\overline{p}},$$

where $\overline{px}$ is the mean of px and $1-\overline{p}$ is the mean $1-p$, i.e., the share of unemployed voters.

Preferences are represented by

$$u(\tau,p,x) = pv((1-\tau)x) + (1-p)v(b),$$

where $v' > 0$, $v'' < 0$ is the utility over consumption.

p IS CONSTANT

We have

$$\begin{aligned}
\frac{\partial^2 u}{\partial \tau \partial x} &= \frac{\partial}{\partial \tau}\left[pv'((1-\tau)x)(1-\tau)\right] \\
&= -pv''((1-\tau)x)(1-\tau)x - pv'((1-\tau))x \\
&= p\left[-\frac{v''((1-\tau)x)(1-\tau)x}{v'((1-\tau)x)} - 1\right] v'((1-\tau)x) \\
&= p\left[\rho_v((1-\tau)x) - 1\right] v'((1-\tau)x),
\end{aligned}$$

where ρ_v is the relative risk aversion.

So, if $\rho_v > 1$ (voters are highly risk averse), $\frac{\partial^2 u}{\partial \tau \partial x} > 0$, and so income x and taxes τ are complements, i.e., a richer voters wants more taxes! (Moene & Wallerstein 2001, APSR)

Intuition: when $\rho_v > 1$, insurance works as a normal good, so higher income leads to higher demand.

Assuming one-dimensional Downs competition, elections implement the median voter's preferred tax rate τ^*.

Recall that $b = \tau \frac{\overline{px}}{1-\overline{p}} = \tau \frac{p}{1-p} \overline{x}$, and

$$u(p, x, \tau, \overline{x}) = pv\left((1-\tau)x\right) + (1-p)v\left(\tau \frac{p}{1-p}\overline{x}\right).$$

We know that τ^* maximizes $u(p, x_{\text{med}}, \tau, \overline{x})$. What happens if inequality, measured by the gap $\overline{x} - x_{\text{med}}$, increases?

Keeping $\overline{x}$ fixed, and assuming $x_{\text{med}} < \overline{x}$, this means decreasing x_{med}. We know that if x decreases, τ decreases.

Hence, more inequality leads to less taxes! This is the opposite of Meltzer-Richard.

Note that the unemployment benefit has a redistributive component. Moene & Wallerstein argue that a large share of social policy in the developed world is a mixture of insurance and redistribution, so the force that we capture in this toy model is more important than the one highlighted by Meltzer-Richard. VOC builds on this insight.

x CONSTANT

We have

$$u(p, x, \tau) = pv\left((1-\tau)x\right) + (1-p)v\left(\tau \frac{\overline{p}}{1-\overline{p}} x\right)$$

and

$$\frac{\partial^2 u}{\partial p \partial \tau} = -v'\left((1-\tau)x\right)x - v'\left(\tau \frac{\overline{p}}{1-\overline{p}} x\right)\frac{\overline{p}}{1-\overline{p}} x < 0$$

so higher p (i.e., lower risk), lower τ.

Again, the voters with median unemployment risk decides policy.

The effect of inequality on unemployment benefit generosity depends on whether $p_{\text{med}} < \overline{p}$ or not. If $p_{\text{med}} > \overline{p}$, fixing $\overline{p}$ we have that if risk inequality increases p_{med} increases (i.e., the median risk decreases) and τ decreases, so b decreases. In other words, more risk inequality, less benefit generosity. This is what Rehm (2011, World Politics) observes in the data.